Some Recurrence Formulas for Box Splines
and
Cone Splines

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Abstract.

A degree elevation formula for multivariate simplex splines was given by Micchelli [6] and extended to hold for multivariate Dirichlet splines in [8]. We report similar formulae for multivariate cone splines and box splines. To this end, we utilize a relation due to Dahmen and Micchelli [4] that connects box splines and cone splines and a degree reduction formula given by Cohen, Lyche, and Riesenfeld in [2].

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1 Introduction

In 1979, C.A. Micchelli reported a formula whereby a multivariate simplex spline of total degree n could be expressed as a linear combination of simplex splines of total degree $n+1$. The knot set of the kth degree $n+1$ spline was formed by adding one more copy of the kth knot to the original knot set. Thus the expansion given in [6] contained many different knot sets.

In [8] the authors generalized the degree elevation formula given by Micchelli to hold for multivariate Dirichlet splines. This class of splines holds as a special case the polynomial simplex splines in addition to rational splines. The formula given in [8] also utilizes the same knot sets as those of [6].

Cohen, Lyche, and Schumaker [3] formulated and solved a degree elevation problem for univariate splines in 1986. Their work illustrated how an arbitrary degree n piecewise polynomial could be constructed from degree $n+1$ splines. In order to preserve regularity at the knot points, they utilized the same knot set for each spline in their expansion.

Recently, we have found some results similar to Micchelli's for box splines and cone splines. It is hoped that the formulae given here may be of some use in formulating and solving a multivariate analog of the problem studied in [3].

The remainder of the paper is follows. Section 2 contains notation, definitions, and basic results necessary to the sequel while the degree elevation formulae for box splines and cone splines are given in Section 3.

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2 Notation, Definitions, and Basic Results

Throughout this paper, we will assume that $n \geq s \geq 1$ are integers and make use of the sets

$$\begin{aligned} \mathbb{R}_+ &= \{x \in \mathbb{R} | x \geq 0\} \\ \mathbb{Z}_+ &= \{n \in \mathbb{Z} | n \geq 0\} \\ I &= [0,1] \end{aligned}$$

and

$$S^{n-1} = \{(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} | \sum_{k=1}^{n-1} t_k \leq 1, t_k \geq 0, k = 1, \ldots, n\}.$$

Often,

$$t_n = 1 - \sum_{k=1}^{n-1} t_k, \tag{2.1}$$

and for $r \in \mathbb{R}$, $r_+ = \max\{0, r\}$. For $X = \{x^1, \ldots, x^n\} \subset \mathbb{R}^s$, other sets of importance are $<X>$, the linear span of the vectors in X, and $[X]$, the convex hull of the vectors in X. Finally, where

$$x^k \in X, t \in \mathbb{R}^n,$$

and for $j = 1, \ldots, n$,

$$\begin{aligned} X_j &= X \backslash \{x^j\}, \\ X^j &= X \cup \{x^j\}. \end{aligned} \tag{2.2}$$

We now define the multivariate splines used throughout the sequel. Let $f \in C_0(\mathbb{R}^s)$ – the space of continuous functions on $\mathbb{R}^s$ with compact support. Assume that X is defined as above with the added restrictions that $0 \notin X$ and $<X> = \mathbb{R}^s$. We define the *box spline*, $B(x|X)$, by requiring the distributional relation

$$\int_{\mathbb{R}^s} f(x) B(x|X) dx = \int_{I^n} f(Xt) dt \tag{2.3}$$

holds. If, in addition $0 \notin [X]$, we define the *cone spline*, $C(x|X)$, by requiring the relation

$$\int_{\mathbb{R}^s} f(x) C(x|X) dx = \int_{\mathbb{R}^n_+} f(Xt) dt \tag{2.4}$$

hold. Here $dx = dx_1 \cdots dx_s$ and $dt = dt_1 \cdots dt_n$.

Now assume $X = \{x^1, \ldots, x^n\} \subset \mathbb{R}^s$ is such that $\mathrm{vol}_s[X] > 0$. Then we define the *simplex spline*, $S(x|X)$ by requiring

$$\int_{\mathbb{R}^s} f(x) S(x|X) dx = (n-1)! \int_{S^{n-1}} f(Xt) dt$$

holds, where now $dt = dt_1 \cdots dt_n$, and t_n is as defined by (2.1).

It should also be noted that the total degree of each spline above depends on the number of vectors in X. Indeed, for $X = \{x^1, \ldots, x^n\}$,

$$\begin{aligned} \deg(B(\cdot|X)) &= n - s, \\ \deg(C(\cdot|X)) &= n - s, \\ \deg(S(\cdot|X)) &= n - 1 - s. \end{aligned}$$

It is also well-known (see [1] for example) that the box spline is compactly supported with

$$\text{spt}(B(\cdot|X)) = \{\sum_{k=1}^{n} t_k x^k \ |t_k \in [0,1], k = 1, \ldots, n\} \tag{2.5}$$

In addition, observe that for $k = 1, \ldots, n$, $\text{spt}(B(\cdot|X)) \subset \text{spt}(B(\cdot|X^k))$.

We conclude Section 2 with two results that illustrate certain relationships between different types of splines. The first result utilizes the following definition:

Let $y \in \mathbb{R}^s$. Then the *backwards difference operator*, $\nabla_y(\cdot)$ is given by:

$$\nabla_y f(\cdot) = f(\cdot) - f(\cdot - y).$$

Also, for $X = \{x^1, \ldots, x^n\}$, we define the operator

$$\nabla_X = \nabla_{x^1} \cdots \nabla_{x^n} .$$

The first result shows that a box spline can be obtained by applying ∇_X to a cone spline with the same knots. The theorem was formulated and proved by W. Dahmen and C.A. Micchelli in [4], and an alternate proof was given by E. Neuman in [7].

THEOREM 2.1 ([4]). *Assume* $X = \{x^1, \ldots, x^n\} \subset \mathbb{R}^s$ *with* $0 \notin [X]$ *and* $<X>= \mathbb{R}^s$. *Then*

$$B(x|X) = \nabla_X C(x|X) \tag{2.6}$$

The final result of Section 2 is due to E. Cohen, T. Lyche, and R. Reisenfeld [2] and is an extension of a result due to Dahmen and Micchelli [5]. The result illustrates how an s-variate cone spline can be evaluated using $(s-1)$-variate simplex splines. The ability to "drop" a dimension is an important tool for the derivation of the degree elevation formulas that are given in Section 3. To state the result of [2], we need to introduce more notation. To this end, we follow [2] and define for any nonsingular matrix $V \in \mathbb{R}^{s \times s}$, a linear functional

$$\pi(x) = e^T V^{-1} x, \tag{2.7}$$

a function

$$\lambda(x) = \begin{cases} \frac{V^{-1}x}{\pi(x)} & \text{if } \pi(x) \neq 0 \\ V^{-1}x & \text{otherwise,} \end{cases}$$

and a projection

$$P(x) = P(x_1, \ldots, x_s) = (x_2, \ldots, x_s)^T.$$

Here, $e^T = (1, \ldots, 1)^T \in \mathbb{R}^s$.

THEOREM 2.2 ([2]). *Let $X = \{x^1, \ldots, x^n\} \subset \mathbb{R}^s$, $n \geq s \geq 2$, with $0 \notin [X]$, $< X >= \mathbb{R}^s$. Then for any nonsingular matrix $V \in \mathbb{R}^{s \times s}$ with $\pi(x^k) > 0$, $k = 1, \ldots, n$, we have for $x \in \mathbb{R}^s$, $\pi(x) \neq 0$,*

$$C(x|X) = \left[(n-1)!|\det(V)| \prod_{k=1}^{n} \pi(x^k)\right]^{-1} \pi(x)_+^{n-s} S(P\lambda(x)|P\lambda(X)), \tag{2.8}$$

where $P\lambda(X) = \{P\lambda(x^1), \ldots, P\lambda(x^n)\}$.

3 Degree Elevation Formulas

In his paper [6], Micchelli gives the following degree elevation formula for multivariate simplex splines:

THEOREM 3.1 ([6]). *Let $X = \{x^1, \ldots, x^n\} \subset \mathbb{R}^s$, and $vol_s[X] > 0$. Then*

$$S(x|X) = \frac{1}{n} \sum_{k=1}^{n} S(x|X^k). \tag{3.1}$$

A generalization of the above result appears in [8]. While the knot set remains constant for the degree elevation formula in [3], the sets utilized in the sequel are the same as those used in Theorem 3.1.

The results that follow deal with degree elevation for cone splines and box splines.

THEOREM 3.2. *Under the assumptions of Theorem 2.2 along with $\pi(x) > 0$,*

$$\pi(x) C(x|X) = \sum_{k=1}^{n} \pi(x^k) C(x|X^k). \tag{3.2}$$

PROOF. Since $\pi(x) \geq 0$, we can insert (3.1) into (2.8) and arrive at

$$C(x|X) = \sum_{k=1}^{n} [n!|\det(V)| \prod_{j=1}^{n} \pi(x^j)]^{-1} \pi^{n-s}(x) S(P\lambda(x)|P\lambda(X^k)). \tag{3.3}$$

Now multiply and divide each term in the sum of (3.3) by $\pi(x^k)$ to obtain:

$$C(x|X) = \sum_{k=1}^{n} \pi(x^k)[n!|\det(V)|\pi(x^k) \prod_{j=1}^{n} \pi(x^j)]^{-1} \pi^{n-s}(x) S(P\lambda(x)|P\lambda(X^k))$$

To complete the proof, multiply both sides of the above equation by $\pi(x)$ and then use (2.8):

$$\begin{aligned}\pi(x)C(x|X) &= \sum_{k=1}^{n} \pi(x^k)[n!|\det(V)|\pi(x^k)\prod_{j=1}^{n}\pi(x^j)]^{-1}\pi^{n+1-s}(x)S(P\lambda(x)|P\lambda(X^k)) \\ &= \sum_{k=1}^{n}\pi(x^k)C(x|X^k).\end{aligned}$$

□

EXAMPLE 3.1. *Let $s = 2$ and $X = \{(1,0)^T, (0,1)^T, (1,1)^T\}$. Figure 3.1 illustrates the cone spline $C(x|X)$ while Figure 3.2 shows the cone splines $C(x|X^k)$, $k = 1, 2, 3$, used in (3.2).*

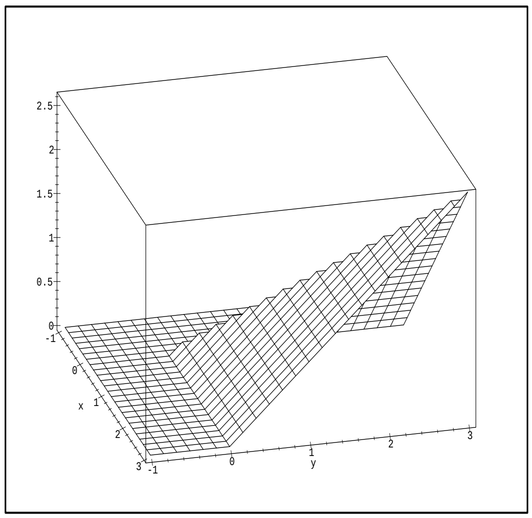

Figure 3.1 The cone spline $C(x|X)$.

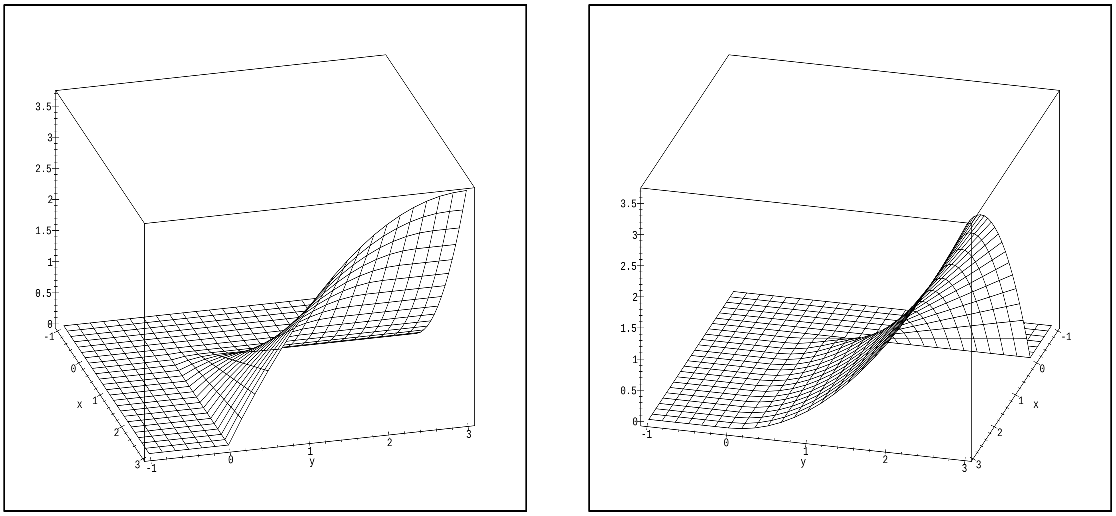

Figure 3.2 The cone splines $C(x|X^1)$ and $C(x|X^2)$.

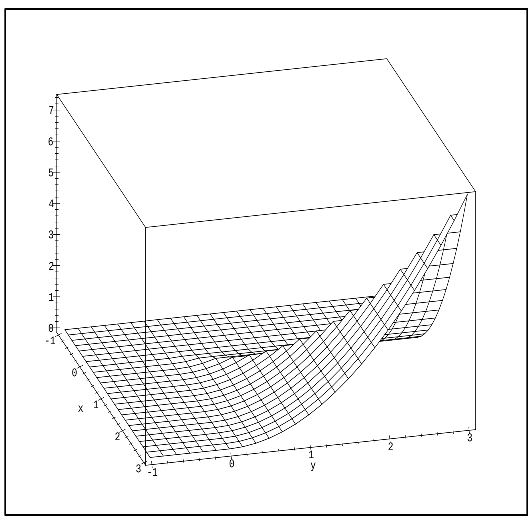

Figure 3.2 (continued) The cone spline $C(x|X^3)$.

In order to prove our result for box splines, we need the following simple lemma.

LEMMA 3.3. *Let* $f, g : \mathbb{R}^s \mapsto \mathbb{R}$ *with* g *linear. Let* $X = \{x^1, \ldots, x^n\} \subset \mathbb{R}^s$.

Then

$$\nabla_X(f(x)g(x)) = g(x)\nabla_X(f(x) + \sum_{k=1}^{n} g(x^k)\nabla_{X_k} f(x - x^k). \tag{3.4}$$

PROOF. The proof follows by induction on n. □

Before stating and proving our final result, let us further analyze the operator ∇_y. Suppose f is continuous and compactly supported with $\operatorname{spt}(f) = S$. Let $M \in \mathbb{N}$ be such that $f(x - \ell y) = 0$ for any $x \in S$ and $\ell > M$. Next consider the convolution product

$$\sum_{j=0}^{M} \delta(x - jy) * \nabla_y f(x) = \sum_{j=0}^{M} \delta(x - jy) * (f(x) - f(x - y)),$$

where δ is the Dirac delta distribution. The fact that f is smooth and compactly supported allows us to formally write

$$\begin{aligned}
\sum_{j=0}^{M} \delta(x - jy) * \nabla_y f(x) &= \sum_{j=0}^{M} \delta(x - jy) * (f(x) - f(x - y)) \\
&= \sum_{j=0}^{M} f(x - jy) - f(x - (j+1)y) \\
&= f(x) - f(x - (M+1)y) \\
&= f(x). \qquad (3.5)
\end{aligned}$$

We shall make use of (3.5) in the proof of our next theorem.

THEOREM 3.4. *Under the assumptions of Theorem 2.2 along with* $\pi(x) > 0$,

$$\pi(x)B(x|X) = \sum_{k=1}^{n} \pi(x^k)(\nabla_X C(x|X^k) - \nabla_{X_k} C(x - x^k|X)). \tag{3.6}$$

Moreover we have

$$\pi(x)B(x|X) = \sum_{k=1}^{n} \pi(x^k) \sum_{j=0}^{M_k} (B(x - jx^k|X^k) + B(x - jx^k|X)), \tag{3.7}$$

For every $x \in spt(B(\cdot|X))$ *where* $B(x|X)$ *is continuous. Here,* $M_k \in \mathbb{N}$, $k = 1, \ldots, n$ *are such that* $B(x - \ell x^k|X^k) = 0$ *whenever* $x \in spt(B(\cdot|X)$, *and* $\ell > M_k$.

PROOF. We first apply ∇_X to both sides of (3.2) to obtain

$$\nabla_X(\pi(x)C(x|X)) = \sum_{k=1}^{n} \pi(x^k)\nabla_X C(x|X^k). \tag{3.8}$$

Applying Lemma 3.3 to the left hand side of (3.8) gives

$$\pi(x)\nabla_X C(x|X) + \sum_{k=1}^{n} \pi(x^k)\nabla_{X_k} C(x - x^k|X) = \sum_{k=1}^{n} \pi(x^k)\nabla_X C(x|X^k).$$

Upon application of Theorem 2.1 (3.6) follows.

In order to establish (3.7), we first apply the operator ∇_X to both sides of (3.6) and next appeal to (2.1). We have

$$\nabla_X(\pi(x)B(x|X)) = \sum_{k=1}^{n} \pi(x^k)(B(x|X^k) - B(x - x^k|X)). \tag{3.9}$$

Now define

$$T_{x^k}(x) = \sum_{j=0}^{M_k} \delta(x - jx^k)$$

where M_k is as described in the theorem. We define the n-fold convolution product

$$T_X = T_{x^n} * \cdots * T_{x^1}$$

and convolve $T_X(x)$ with both sides of (3.9) to obtain

$$T_X(x) * (\pi(x)B(x|X)) = \sum_{k=1}^{n} \pi(x^k)T_X(x) * (B(x|X^k) - B(x - x^k|X)). \tag{3.10}$$

Equation (3.5) then gives

$$\begin{aligned} \pi(x)B(x|X) &= \sum_{k=1}^{n} \pi(x^k)T_{x^k} * (B(x|X^k) - B(x - x^k|X)) \\ &= \sum_{k=1}^{n} \pi(x^k)\delta(x - jx^k) * (B(x|X^k) - B(x - x^k|X)) \\ &= \sum_{k=1}^{n} \pi(x^k) \sum_{j=0}^{M_k}(B(x - jx^k|X^k) - B(x - (j+1)x^k|X)) \end{aligned}$$

from which the result follows. □

We conclude the paper with an example of the elevation formula for box splines.

EXAMPLE 3.1. *Let $s = 2$ and $X = \{(1,0)^T, (0,1)^T, (1,1)^T\}$. Figure 3.3 shows the bivariate box spline $B(x|X)$ while Figure 3.4 shows the box splines $B(x|X^k)$, $k = 1, 2, 3$ who along with their translates and the translates of $B(x|X)$ are used in (3.7) to obtain $B(x|X)$.*

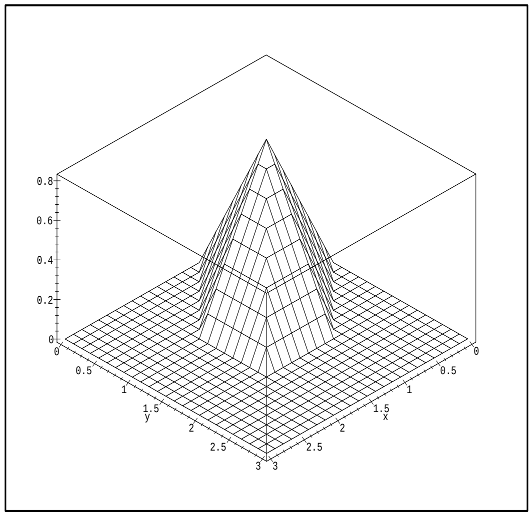

Figure 3.3 The box spline $B(x|X)$.

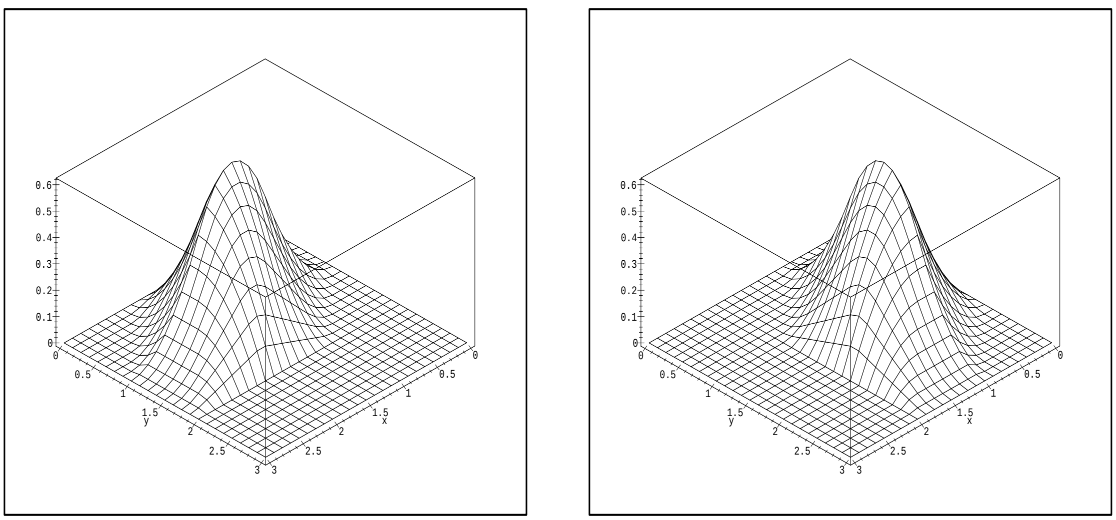

Figure 3.4 The box splines $B(x|X^1)$ and $B(x|X^2)$.

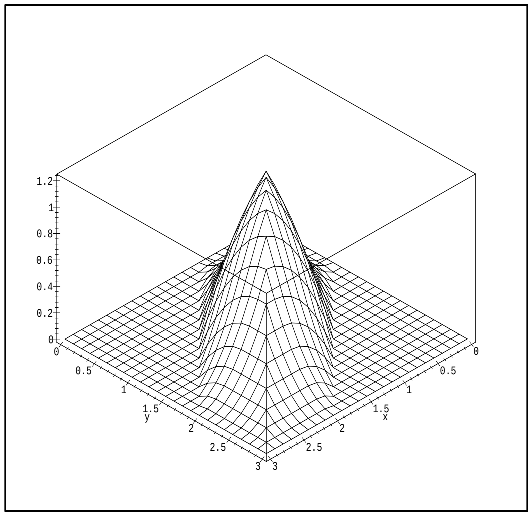

Figure 3.4 (continued) The box spline $B(x|X^3)$.

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