Gibbs' Phenomenon for Nonnegative Compactly Supported Scaling Vectors

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Abstract

This paper considers Gibbs' phenomenon for scaling vectors in $L^2(\mathbb{R})$. We first show that a wide class of multiresolution analyses suffer from Gibbs' phenomenon.

To deal with this problem, in [11], Walter and Shen use an Abel summation technique to construct a positive scaling function P_r, $0 < r < 1$, from an orthonormal scaling function ϕ that generates V_0. A reproducing kernel can in turn be constructed using P_r. This kernel is also positive, has unit integral, and approximations utilitizing it display no Gibbs' phenomenon. These results were extended to scaling vectors and multiwavelets in [9]. In both cases, orthogonality and compact support were lost in the construction process.

In this paper we modify the approach given in [9] to construct compactly supported positive scaling vectors. While the mapping into V_0 associated with this new positive scaling vector is not a projection, the scaling vector does produce a Riesz basis for V_0 and we conclude the paper by illustrating that expansions of functions via positive scaling vectors exhibit no Gibbs' phenomenon.

Key words: scaling functions, scaling vectors, Gibbs' phenomenon, summability techniques, compactly supported scaling vectors

1 Introduction

We consider the question of Gibbs' phenomenon for scaling vector expansions. Generalizing a result of Shim and Volkmer [10], we show that if Φ is orthogonal or Φ has a biorthogonal dual that is compactly supported, then the corresponding wavelet expansion exhibits Gibbs' phenomenon on at least

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one side of 0. The question of how to avoid Gibbs' for wavelet expansions is thus important, and was first studied by Walter and Shen [11].

Let ϕ be a compactly supported orthogonal scaling function generating a multiresolution analysis $\{V_k\}$ for $L^2(\mathbb{R})$. In [11], the authors show how to use this ϕ to construct a new scaling function P that generates the same multiresolution analysis for $L^2(\mathbb{R})$. Moreover, $P(t) \geq 0$ for $t \in \mathbb{R}$. The application the authors considered for this new function P was density estimation. They also showed that approximations $f_m \in V_m$ to $f \in L^2(\mathbb{R})$ where $f_m(t) = \int_{s\in\mathbb{R}} K_m(s,t)f(s)ds$ and $K_m(s,t) = 2^m \sum_{n\in\mathbb{Z}} \phi(2^m s - n)\phi(2^m t - n)$ exhibits no Gibbs' phenomenon. While K_m is not a projection of f into V_m, f_m may well be useful in some applications where Gibbs' phenomenon is a problem. The disadvantages of this construction are that P is not compactly supported and orthogonality is lost (although the authors gave a simple expression for the dual P^*).

The results of Walter and Shen [11] were generalized to the scaling vectors $\Phi = (\phi^1, \ldots, \phi^A)^T$ in [9]. Here the authors also showed that it was not necessary to start with an orthogonal scaling vector supported on some interval $[0, M]$ to construct the nonnegative scaling vector P.

While the orthogonality of a scaling vector is desirable in some cases, it is impossible to insist that the scaling vector be both orthogonal and nonnegative. As we will see, is it often possible to modify the construction and retain the compact support. We will take a bounded, compactly supported scaling vector Φ and illustrate how to construct a nonnegative compactly supported scaling vector $\tilde{\Phi}$ that generates the same multiresolution analysis as Φ. The construction requires that at least one component ϕ^j of Φ is nonnegative on its support plus some conditions on the coefficients in the partition of unity generated by Φ. We then prove that Gibbs' is avoided by the new scaling vector, and the results are applied to two well-known scaling vectors from the literature.

2 Notation, Definitions, and Preliminary Results

In this section we will state definitions, introduce notation, and present results used throughout the sequel.

We begin with the concept of a *scaling vector* or a set of multiscaling functions. This idea was first introduced in [3,5]. We start with A functions, $\phi^1, \ldots, \phi^A$ and consider the space $V_0 = \overline{< \{\phi^1(\cdot - k), \ldots, \phi^A(\cdot - k)\}_{k\in\mathbb{Z}} >}$.

It is convenient to store $\phi^1, \ldots, \phi^A$ in a vector $\Phi(t) = \begin{pmatrix} \phi^1(t) & \phi^2(t) & \ldots & \phi^A(t) \end{pmatrix}^T$ and define a *multiresolution analysis* in much the same manner as in [1]:

- (M1) $\overline{\cup_{n\in\mathbb{Z}}V_n} = L^2(\mathbb{R})$.
- (M2) $\cap_{n\in\mathbb{Z}}V_n = \{0\}$.
- (M3)$f \in V_n \leftrightarrow f(2^{-n}\cdot) \in V_0$, $n \in\mathbb{Z}$.
- (M4) $f \in V_0 \rightarrow f(\cdot - n) \in V_0$, $n \in\mathbb{Z}$.
- (M5) Φ generates a Riesz basis for V_0.

In this case Φ satisfies a *matrix refinement equation*:

$$\Phi(x) = \sum_{k=0}^{N} C_k \Phi(2x-k) \tag{1}$$

where the C_k are $A \times A$ matrices. We define the *Fourier transform* $\hat{\Phi}$ of Φ by the component-wise rule: $\hat{\phi}^\ell(\omega) = \int_{\mathbb{R}} \phi^\ell(t)e^{-i\omega t}dt,\ \ell = 1, \ldots, A$ and the $A \times A$ matrix

$$E_\Phi(\omega) = \sum_{k\in\mathbb{Z}} \hat{\Phi}(\omega + 2\pi k)\hat{\Phi}^\dagger(\omega + 2\pi k) \tag{2}$$

where $\dagger$ denotes the Hermitian conjugate. The matrix E_Φ plays an important role in analyzing scaling vectors. Indeed Geronimo, Hardin, and Massopust introduced this matrix in [3] and showed that the nonsingularity of E_Φ is necessary and sufficient for the set in (M5) to form a Riesz basis for V_0.

We introduce standard terminology: Φ is *continuous (bounded)* if each component function ϕ^ℓ is continuous (bounded). Similarly, Φ has *compact support* if each component function ϕ^ℓ is compactly supported. In this case, we assume that $\text{supp}(\phi^\ell) = [0, M_\ell]$ and denote by M the maximum value of M_ℓ:

$$M = \max\{M_1, \ldots, M_A\} \tag{3}$$

We will say that Φ has *polynomial accuracy* p if $t^k \in V_0$ for $k = 0, 1, \ldots, p-1$. In particular, for the case $p = 1$ (partition of unity), this is equivalent to the existence of a vector $\vec{c} = (c_1, \ldots, c_A)^T$ for which

$$\sum_{\ell=1}^{A} \sum_{k\in\mathbb{Z}} c_\ell \phi^\ell(t-k) = 1. \tag{4}$$

It was shown by Theorem 3.1 in [9] that if Φ is a continuous, compactly supported scaling vector with accuracy $p \geq 1$ satisfying (M1)-(M5), a new scaling vector $\tilde{\Phi}$ could be constructed that generates the same multiresolution analysis as Φ and also satisfies:

- $\sum_{k\in\mathbb{Z}} \phi^\ell(t-k) > 0$ for each $\ell \in\mathbb{Z}$ such that $c_\ell \neq 0$, and
- $c_\ell \int_{\mathbb{R}} \phi^\ell \geq 0$ for each $\ell = 1, \ldots, A$ and if $\int_{\mathbb{R}} \phi^\ell = 0$, then $c_\ell = 0$.

This new scaling vector $\tilde{\Phi}$ could then be used to construct a kernel allowing one to avoid Gibbs' (Proposition 3.7 in [9]). However, compact support was lost in the construction of this new scaling vector $\tilde{\Phi}$. In Section 4, we will show how to construct a new scaling vector $\tilde{\Phi}$ preserving the compact support, and which is used to construct a kernel allowing one to avoid Gibbs'.

3 Gibbs' Phenomenon for Nonnegative Scaling Vectors

In this section, we prove a theorem demonstrating that Gibbs' phenomenon is indeed a problem for a wide class of multiresolution analyses such as those found in [4], [3] and others. To clarify the discussion, we classify multiresolution analyses into three categories:

(MRA1) Those with orthonormal bases. In this case we can write

$$L^2(\mathbb{R}) = V_k \oplus (\oplus_{\ell \geq k} W_\ell)$$

where the direct sums are orthogonal, and the corresponding orthogonal projections P_k are defined by

$$P_k \left(\sum_{i=1}^{A} \sum_{j \in \mathbb{Z}} \alpha_{kj}^i \phi_{kj}^i + \sum_{\ell \geq k} \sum_{i=1}^{A} \sum_{j \in \mathbb{Z}} \beta_{\ell j}^i \psi_{\ell j}^i \right) = \sum_{ij} \alpha_{kj}^i \phi_{kj}^i \tag{5}$$

where $\phi_{kj}^i(t) = 2^{-k/2} \phi^i(2^k t - j) \quad$ for $i = 1, \ldots, A, \quad k, j \in \mathbb{Z}$.

(MRA2) Those with semi-orthogonal bases. In this case the translates of the scaling function(s) are not orthogonal, but we can still write

$$L^2(\mathbb{R}) = V_k \oplus (\oplus_{\ell \geq k} W_\ell)$$

where the direct sums are orthogonal, and the corresponding P_k are defined as in (5).

(MRA3) Those with non-orthogonal biorthogonal bases. In this case the V_j and W_j spaces are non-orthogonal and

$$L^2(\mathbb{R}) = V_k \bigoplus \left(\bigoplus_{\ell \geq k} W_\ell \right)$$

where the direct sums $\bigoplus$ are not orthogonal, and the corresponding P_k defined as in (5) are not orthogonal. In this case, there is a dual multiresolution analysis with scaling vector Φ^* such that $< \phi_{kj}^i, {\phi^*}_{mn}^{\ell} >= \delta_{i\ell} \delta_{km} \delta_{jn}$, $k, j, m, n \in \mathbb{Z}$, $i, \ell = 1, \ldots, A$.

Here is a precise definition of Gibbs' phenomenon.

Definition 3.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a square integrable bounded function with a jump discontinuity at* 0*: the limits* $\lim_{x\to 0^+} f(x) = f(0+)$ *and* $\lim_{x\to 0^-} f(x) = f(0-)$ *exist and are different. Without loss of generality we assume* $f(0+) > f(0-)$*. Suppose we have a multiresolution analysis of* $L^2(\mathbb{R})$ *with multiresolution spaces* (V_j) *generated by a scaling vector. We say a sequence of operators* (L_j)*,* $L_j : L^2(\mathbb{R}) \to V_j$ *is* **admissible** *if* $\lim_{j\to\infty} L_j(f) = f$ *in the* L^2 *sense, for all* $f \in L^2(\mathbb{R})$*. We say that a wavelet expansion of* f *with respect to a scaling vector and an admissible sequence* (L_j) *shows a Gibbs' phenomenon at* 0 *if there is a positive sequence* (x_m) *with* $\lim_{m\to\infty} x_m = 0$ *and* $\lim_{m\to\infty} L_m(f(x_m)) > f(0+)$*, or if there is a negative sequence* (t_m) *with* $\lim_{m\to\infty} t_m = 0$ *and* $\lim_{m\to\infty} L_m(f(t_m)) < f(0-)$*.*

Observe that we do not require the maps L_j to be orthogonal projections since many interesting MRA's are built from Riesz or biorthogonal bases, rather than orthogonal bases. Moreover, we shall see that we can avoid Gibbs' phenomenon by taking an admissible sequence of operators that are not even projections. The definition is otherwise quite standard. Our main result is to show that nearly all interesting scaling vectors generating multiresolution analyses will suffer from Gibbs' phenomenon. More precisely, we prove the theorem below.

Theorem 3.2 *Let* $\Phi = (\phi^1, \ldots, \phi^A)^T$ *be a continuous, compactly supported scaling vector with polynomial accuracy at least* 2*. If the multiresolution analysis is orthogonal or* Φ *has a dual biorthogonal basis* Φ^* *that is compactly supported, then the corresponding wavelet expansion shows a Gibbs' phenomenon at least one side of* 0*.*

To prove this result, we modify and generalize Shim and Volkmer's [10] approach for the single scaling function orthonormal case in two directions: to include biorthogonal bases and to include multiple scaling functions. We are also able to replace a pair of rather technical derivative and decay hypotheses in [10] with the hypotheses on compact support and polynomial accuracy. We now state their main result from [10].

Theorem 3.3 (Shim, Volkmer) *Let* ϕ *be a continuous scaling function generating an orthonormal multiresolution analysis that is differentiable at a dyadic number with a nonvanishing derivative there, and that satisfies*

$$|\phi(t)| \leq K(1 + |t|)^{-\beta} \text{for } t \in \mathbb{R}$$

with constants $K > 0$ *and* $\beta > 3$*. Then the corresponding wavelet expansion shows a Gibbs' phenomenon at one side of* 0*.*

Before we present the proof to Theorem 3.2, we first introduce some notation and state and prove two lemmas. Let Q_m denote the projection map onto the space V_m defined above in (5). Define the reproducing kernel $q(s,t)$ by

$$q(s,t) = \sum_{i=1}^{A} \sum_{j \in \mathbb{Z}} \phi^i(s-j)\phi^{*i}(t-j) \tag{6}$$

and q_m by $q_m(s,t) = 2^m q(2^m s, 2^m t)$, where (ϕ^{*i}) is the biorthogonal basis. Observe that

$$(Q_0 f)(s) = \sum_{i=1}^{A} \sum_{j \in \mathbb{Z}} \left\langle f, \phi^{*i}(\cdot - j) \right\rangle \phi^i(s-j) = \int_{\mathbb{R}} f(t) q(s,t) dt$$

$\forall f \in L^2(\mathbb{R})$. Finally, let

$$H(t) = \begin{cases} 1 \text{ if } t > 0 \\ -1 \text{ if } t < 0 \end{cases}$$

and define function r by $r = H - Q_0 H$.

Lemma 3.4 *The coefficients c_i in (4) satisfy $c_i = \int_{\mathbb{R}} \phi^{*i}(t)dt$ and $\int_{\mathbb{R}} q_m(s,t)dt = 1$ for $m \in \mathbb{Z}$.*

Proof. First observe that from the biorthogonality and (4), we have

$$\int_{\mathbb{R}} \phi^{*i}(t)dx = \int_{\mathbb{R}} \phi^{*i}(t) \sum_{\ell=1}^{A} \sum_{k \in \mathbb{Z}} c_\ell \phi^\ell(t-k)dt = c_i$$

The second result follows from integrating (6) with respect to t and applying our formula for c_i and (4). $\square$

Lemma 3.5 *Let $\Phi = (\phi^1, \ldots, \phi^A)^T$ be a compactly supported, continuous scaling vector with accuracy $p \geq 2$ generating a multiresolution analysis for $L^2(\mathbb{R})$. If the multiresolution analysis is orthogonal or Φ has a dual biorthogonal basis Φ^* that is compactly supported then the following are true:*

(1) $Q_0 H = H - r$ is continuous,
(2) $r(t)$ is compactly supported and continuous, except for a jump discontinuity at 0.
(3) $r \in \bigoplus_{j \geq 0} W_j^*$
(4) $\int_{\mathbb{R}} tr(t)dt = 0$.

Proof.

1. First note that $(Q_0H)(s) = \int_{\mathbb{R}} H(t)q(s,t)dt = \sum_{n,\ell} \phi^\ell(s-n)d_{n,\ell}$ where

$$d_{n,\ell} = \int_0^\infty \phi^{*\ell}(t-n)dt - \int_{-\infty}^0 \phi^{*\ell}(t-n)dt.$$

Each $\phi^\ell(\cdot - n)$ is continuous and compactly supported, so Q_0H is continuous.

2. r is continuous except for a jump discontinuity at 0. This follows from Part 1 and the fact that $r = H - Q_0H$. Thus it suffices to show that r has compact support. To this end, observe that for $t \geq 0$, Lemma 3.4 tells us that

$$r(t) = 1 - \int_{\mathbb{R}} q(t,y)H(y)dy = 2\int_{-\infty}^0 q(t,y)dy$$

Similarly for $t < 0$, $r(t) = -2\int_0^\infty q(t,y)dy$. Now by the compact support of the ϕ^ℓ and $\phi^{*\ell}$, for $t > M$, where M is given by (3) we have

$$r(t) = 2\sum_{\ell=1}^{A}\sum_{n\geq 0} \phi^\ell(t-n)\int_{-\infty}^0 \phi^{*\ell}(y-n)dy = 0.$$

Let M^* be defined by (3) for the dual scaling vector Φ^*. Then for $t < -M^* - M$

$$r(t) = -2\sum_{\ell=1}^{A}\sum_{n=-\infty}^{-M-M^*} \phi^\ell(t-n)\int_0^\infty \phi^{*\ell}(y-n)dy = 0,$$

whence $r(t)$ has compact support.

3. Next, for arbitrary $j = 1, \ldots, A$ and $k \in \mathbb{Z}$, observe that

$$\begin{aligned}\int_{\mathbb{R}} r(t)\phi^{*j}(t-k)dt &= \int_{\mathbb{R}} H(t)\phi^{*j}(t-k)dx - \int_{\mathbb{R}}(Q_0H)(t)\phi^{*j}(t-k)dt \\ &= \int_{\mathbb{R}} H(t)\phi^{*j}(t-k)dt - \int_{\mathbb{R}}\int_{\mathbb{R}}(H(y)q(t,y)dy)\phi^{*j}(t-k)dt\end{aligned}$$

which can be expressed as

$$\begin{aligned}&= \int_{\mathbb{R}} H(t)\phi^{*j}(t-k)dt - \int_{\mathbb{R}}\int_{\mathbb{R}} H(y)\sum_{m=1}^{A}\sum_{n\in\mathbb{Z}}\left[\phi^m(t-n)\phi^*_m(y-n)\right]\phi^{*j}(t-k)dydt \\ &= \int_{\mathbb{R}} H(t)\phi^{*j}(t-k)dt - \int_{\mathbb{R}} H(y)\sum_{m=1}^{A}\sum_{n\in\mathbb{Z}}(\int_{\mathbb{R}}\phi^m(t-n)\phi^{*j}(t-k)dt)\phi^*_m(y-n)dy \\ &= \int_{\mathbb{R}} H(t)\phi^{*j}(t-k)dt - \int_{\mathbb{R}} H(y)\cdot\phi^{*j}(y-k)dy\end{aligned}$$

so $r \perp \Phi^*_{jk}$ $j = 1, \ldots, A$ and $k \in \mathbb{Z}$. Writing $L^2(\mathbb{R}) = V_0^* \oplus \left(\bigoplus_{k\geq 0} W_k^*\right)$ we must have $r \in \bigoplus_{k\geq 0} W_k^*$.

4. Part 3 tells us that $r = \sum_{\ell \geq 0} \alpha_{i,\ell j} \psi^*_{i,\ell j}$ where the $\psi^*_{i,\ell j} \in W^*_\ell$ are the multiwavelets of the dual basis. Since Φ has polynomial accuracy at least 2, $t = \sum_{n,\ell} \beta_{n\ell} \phi^\ell (t-n)$ for some $(\beta_{n\ell})$ so

$$\int_{\mathbb{R}} tr(t)dt = \int_{\mathbb{R}} \left\{ \sum_{n,\ell} \beta_{n\ell} \phi^\ell(\cdot - n) \right\} \left\{ \sum \alpha^l_{ij} \psi^\ell_{ij} \right\} = 0$$

since $V_0 \perp W^*_j$ for each $j \geq 0$, $j \in \mathbb{Z}$.

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. We first claim that $r(t_1) < 0$ for some $t_1 > 0$ or $r(t_2) > 0$ for some $t_2 < 0$. For otherwise $\int_{\mathbb{R}} tr(t)dt = 0$ would force $r(t) = 0$ almost everywhere. This is impossible by Part 2 of Lemma 3.5. Now consider the case $r(t_1) < 0$ for some $t_1 > 0$. Then $r(t_1) = 1 - \int_{\mathbb{R}} q(t_1, y)H(y)dy < 0$ implies that

$$\int_{\mathbb{R}} q(t_1, y)H(y)dy > 1. \tag{7}$$

We now show there must be a Gibbs' phenomenon for the Haar wavelet

$$h(t) = \begin{cases} 1 \text{ if } & 0 \leq t \leq 1 \\ -1 \text{ if } & -1 \leq t < 0 \end{cases}$$

Clearly $\lim_{m\to\infty} t_1 2^{-m} = 0$, but

$$\begin{aligned}
\lim_{m\to\infty} (Q_m h)(t_1 2^{-m}) &= \lim_{m\to\infty} \int_{\mathbb{R}} 2^m q(t_1, 2^m y)h(y)dy \\
&= \lim_{m\to\infty} \int_0^1 2^m q(t_1, 2^m y)dy - \int_{-1}^0 2^m q(t_1, 2^m y)dy \\
&= \lim_{m\to\infty} \int_0^{2^m} q(t_1, t)dt - \int_{-2^m}^0 q(t_1, t)dt \\
&= \int_{-\infty}^{\infty} q(t_1, t)H(t)dt > 1
\end{aligned}$$

by (7). Thus h exhibits Gibbs' phenomenon at 0. The case $r(t_2) > 0$ for some $t_2 < 0$ is similar. □

4 Positive Scaling Vectors with Compact Support

In this section we describe a procedure for constructing compactly supported positive scaling vectors that avoid Gibbs' phenomenon. The idea is to start with a bounded, compactly supported scaling vector Φ with accuracy $p \geq 1$, with the additional requirements that at least one of components ϕ^j of Φ

is nonnegative, plus some conditions on the coefficients in (4). Theorem 4.1 below shows how to transform this scaling vector into a new compactly supported nonnegative scaling vector satisfying the following condition regarding its coefficients in (4).

(A) If $c_k \neq 0$ then $\phi^k(x) \geq 0 \ \forall x \in \mathbb{R}$ and $c_k > 0$.

This new scaling vector satisfying (A) will then be used to construct a kernel allowing one to avoid Gibbs' phenomenon in Theorem 4.3 below. We will complete the paper with two examples demonstrating the results.

Theorem 4.1 *Suppose a scaling vector* $\Phi = (\phi^1, \ldots, \phi^A)^T$ *is bounded, compactly supported, has accuracy* $p \geq 1$*, and satisfies:*

Condition B. *Assume* $\phi^j(x) \geq 0 \ \forall x \in R$ *for some* j *and there exist finite index sets* Λ_i *and constants* g_{ik} *for* $i \neq j$ *such that:*

B1. $\tilde{\phi}^i(t) := \phi^i(t) + \sum_{k \in \Lambda_i} g_{ik} \phi^j(t-k) \geq 0 \ \forall x \in \mathbb{R}$,

B2. $d_j := c_j - \sum_{i \neq j} \sum_{k \in \Lambda_i} c_i g_{ik} \geq 0$,

B3. $c_i \geq 0$ *for* $i \neq j$

where the c_i *are the coefficients in (4) for* Φ.

Then the nonnegative vector $\tilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^{j-1}, \phi^j, \tilde{\phi}^{j+1}, \ldots, \tilde{\phi}^A)^T$ *is a bounded, compactly supported scaling vector with accuracy* $p \geq 1$ *that satisfies (A) and generates the same space* V_0 *as* Φ.

Proof. $\tilde{\Phi}$ is nonnegative, bounded, and compactly supported by the support and boundedness properties of Φ and the assumptions of Condition B.

To prove $\tilde{\Phi}$ satisfies (A) and generates a partition of unity, we start by solving B1 for $\phi^i(t)$ and substituting this into the original partition of unity *(4)*:

$$\sum_{n \in \mathbb{Z}} \left\{ \sum_{i \neq j} c_i \left[\tilde{\phi}^i(t-n) - \sum_{k \in \Lambda_i} g_{ik} \phi^j(t-k-n) \right] + c_j \phi^j(t-n) \right\} = 1$$

so that

$$\sum_{n \in \mathbb{Z}} \sum_{i \neq j} c_i \tilde{\phi}^i(t-n) - \sum_{i \neq j} \sum_{k \in \Lambda_i} c_i g_{ik} \sum_{n \in \mathbb{Z}} \phi^j(t-k-n) + \sum_{n \in \mathbb{Z}} c_j \phi^j(t-n) = 1.$$

Substituting $m = n + k$ into the second expression gives:

$$\sum_{n \in \mathbb{Z}} \sum_{i \neq j} c_i \tilde{\phi}^i(t-n) - \sum_{i \neq j} \sum_{k \in \Lambda_i} c_i g_{ik} \sum_{m \in \mathbb{Z}} \phi^j(t-m) + \sum_{n \in \mathbb{Z}} c_j \phi^j(t-n) = 1$$

or

$$\sum_{n\in\mathbb{Z}}\sum_{i\neq j} c_i\tilde{\phi}^i(t-n) + \sum_{n\in\mathbb{Z}}\left\{c_j - \sum_{i\neq j}\sum_{k\in\Lambda_i} c_i g_{ik}\right\}\phi^j(t-n) = 1.$$

Since $\tilde{\phi}^j = \phi^j$, we get the partition of unity

$$\sum_{i=1}^{A}\sum_{n\in\mathbb{Z}} d_i\tilde{\phi}^i(t-n) = 1$$

where $d_i = c_i \geq 0$ for $i \neq j$ by assumption B3, and

$$d_j = c_j - \sum_{i\neq j}\sum_{k\in\Lambda_i} c_i g_{ik}$$

which is nonnegative by assumption B2. This also shows that (A) holds for $\tilde{\Phi}$.

To see that $\tilde{\Phi}$ forms a Riesz basis for V_0, assume without loss of generality that $j = A$ and note that

$$\hat{\tilde{\Phi}}(\omega) = B(\omega)\hat{\Phi}(\omega)$$

where $B(\omega)$ is an $A \times A$ upper triangular matrix defined by

$$B(\omega) = \begin{bmatrix} I_{A-1} & \vec{m} \\ \vec{0} & 1 \end{bmatrix}$$

where I_{A-1} is the $A-1 \times A-1$ identity matrix, $\vec{0}$ is a $A-1$ row vector of 0's, and $\vec{m}$ is an $A-1$ column vector whose components m_i, $i = 1, \ldots, A-1$, are given by

$$m_i = \sum_{k\in\Lambda_i} g_{ik}e^{-ik\omega},$$

We compute the $A \times A$ matrix

$$\begin{aligned}
E_{\tilde{\Phi}}(\omega) &= \sum_{k\in\mathbb{Z}} \hat{\tilde{\Phi}}(\omega + 2\pi k)\hat{\tilde{\Phi}}^{\dagger}(\omega + 2\pi k) \\
&= \sum_{k\in\mathbb{Z}} B(\omega + 2\pi k)\hat{\Phi}(\omega + 2\pi k)\hat{\Phi}^{\dagger}(\omega + 2\pi k)B^{\dagger}(\omega + 2\pi k) \\
&= B(\omega)\left(\sum_{k\in\mathbb{Z}} \hat{\Phi}(\omega + 2\pi k)\hat{\Phi}^{\dagger}(\omega + 2\pi k)\right)B^{\dagger}(\omega) \\
&= B(\omega)E_{\Phi}(\omega)B^{\dagger}(\omega)
\end{aligned}$$

By definition $B(\omega)$ is nonsingular, so that $B^{\dagger}(\omega)$ is also nonsingular. Since Φ forms a Riesz basis for V_0, we have that $E_{\Phi}(\omega)$ is also nonsingular. Thus $E_{\tilde{\Phi}}(\omega)$ is nonsingular and thus by virtue of Theorem 3.2 in [3], $\tilde{\Phi}$ generates a Riesz basis for V_0. Moreover, $\tilde{\Phi}$ must have the same accuracy $p \geq 1$ as Φ.

We must finally show that $\tilde{\Phi}$ satisfies a matrix refinement equation. Let

$$B = \begin{bmatrix} I_{A-1} & \overrightarrow{g} \\ \vec{0} & 1 \end{bmatrix}$$

where I_{A-1} and $\vec{0}$ are as defined above and the components g_i, $i = 1, \ldots, A-1$ of $\overrightarrow{g}$ are given by

$$g_i = \sum_{k \in \Lambda_i} g_{ik}.$$

Then

$$\tilde{\Phi}(t) = B\Phi(t) = \sum_k BC_k \Phi(2t-k).$$

But B is nonsingular so that we can write

$$\Phi(2t-k) = B^{-1}\tilde{\Phi}(2t-k)$$

and thus observe that the refinement equation coefficients for $\tilde{\Phi}$ are

$$\tilde{C}_k = BC_kB^{-1} \ \square.$$

Remark. A sufficient condition on ϕ^j for the existence of these index sets for Condition B1 is $\phi^j > 0$ on an interval J, where $\bar{J} = [a, b]$ and $b - a \geq 1$.

We next show that we can avoid Gibbs' by using a special reproducing kernel. Of course, the reproducing kernel here corresponds to map into V_m that is not a projection. Note that in Theorem 4.3 below the compact support and positivity together allow a improved statement over our previous result (Proposition 3.7 of [9]) and that of Shen and Walter (Proposition 4.3 of [11]): we can specify the resolution of the kernel and can give a tighter upper bound on the approximation in V_m. While we require the positivity Condition B, we do not need the continuity assumption required in the propositions of [9], [11] just mentioned.

We first define the reproducing kernel

$$K(s,t) = \sum_{c_j \neq 0} \sum_{k \in \mathbb{Z}} \left(\frac{c_j}{\int_{\mathbb{R}} \phi^j} \right) \phi^j(t-k)\phi^j(s-k).$$

For the sake of notation, we define $K_m(s,t)$ by

$$K_m(s,t) = 2^m K(2^m s, 2^m t).$$

Before proving the theorem indicating the absence of Gibbs', we establish some key facts about the kernel K.

Proposition 4.2 *If the bounded, compactly supported scaling vector* Φ *with accuracy* $p \geq 1$ *satisfies (A), then*

(1) $\int_{\mathbb{R}} K_m(s,t)ds = 1 \qquad \forall m \in \mathbb{Z}, t \in \mathbb{R}$

(2) $K_m(s,t) \geq 0 \qquad \forall m \in \mathbb{Z}, t \in \mathbb{R}$

(3) For each $\gamma > 0$, *if* $m > \log_2(\frac{M}{\gamma})$ *then* $\sup_{|s-t|>\gamma} K_m(s,t) = 0$.

Proof. The proof of 1. follows from (4):

$$\int_{\mathbb{R}} K_m(s,t)ds = 2^m \sum_{c_j \neq 0} \sum_{k \in \mathbb{Z}} \left(\frac{c_j}{\int_{\mathbb{R}} \phi^j} \right) \phi^j(t-k)\phi^j(s-k) \int_{\mathbb{R}} \phi^j(2^m s - k)\, ds$$
$$= \sum_{c_j \neq 0} \sum_{k \in \mathbb{Z}} c_j \phi^j(t-k) = \sum_{j=1}^{A} \sum_{k \in \mathbb{Z}} c_j \phi^j(t-k) = 1$$

The proof of (2) follows directly from (A).

To see 3., observe that $|\mathrm{supp}\,(\phi^j(2^m \cdot -k))\,| \leq M2^{-m} < \gamma$ where M is defined in (3). So if $|t-s| > \gamma$ then $\phi^j(2^m s - k)\phi^j(2^m t - k) = 0 \; \forall k \in \mathbb{Z}$. Thus

$$\sup_{|s-t|>\gamma} K_m(s,t) = 2^m \sum_{c_j \neq 0} \sum_{k \in \mathbb{Z}} \left(\frac{c_j}{\int_{\mathbb{R}} \phi^j} \right) \sup_{|s-t|>\gamma} \phi^j(2^m s - k)\phi^j(2^m t - k) = 0. \qquad \square$$

Theorem 4.3 *Let* $\Phi = (\phi^1, \ldots, \phi^A)^T$ *be a bounded, compactly supported scaling vector with accuracy* $p \geq 1$ *satisfying (A). Suppose that* $M_1 \leq f(t) \leq M_2$ *on* $[a,b]$. *Then for each* $\delta > 0$ *and* $m > \log_2\left(\frac{M}{\delta}\right)$,

$$M_1 \leq f_m(t) \leq M_2$$

whenever $t \in (a+\delta, b-\delta)$. *Here,* $f_m \in V_m$ *where*

$$f_m(t) = \int_{\mathbb{R}} K_m(s,t)f(s)ds.$$

Proof. For $t \in (a+\delta, b-\delta)$ choose $m > \log_2\left(\frac{M}{\delta}\right)$ and write $f_m(t)$ as

$$\begin{aligned} f_m(t) &= \int_{\mathbb{R}} K_m(s,t)f(s)ds = \left(\int_{-\infty}^{a} + \int_{a}^{b} + \int_{b}^{\infty} \right) K_m(s,t)f(s)ds \\ &\leq 2 \sup_{|s-t|>\delta} K_m(s,t) \int_{\mathbb{R}} |f(s)|ds + M_2 \int_{\mathbb{R}} K_m(s,t)ds \\ &= M_2 \end{aligned}$$

using the Proposition 4.2 above. The proof that $M_1 \leq f_m(t)$ is similar. $\square$

We conclude by giving two examples that illustrate the results of Theorems 4.1 and 4.3. The first example involves the scaling vector of Donovan, Geronimo, Hardin, and Massopust [2] while the second utilizes the vector constructed by Plonka and Strela [8].

Example 4.4 *In [2], the authors constructed a continuous, orthogonal, symmetric scaling vector that satisfies the matrix refinement equation* $\Phi(t) = \sum_{k=0}^{3} C_k \Phi(2t-k)$ *where* $C_0 = \begin{bmatrix} 3/5 & 4\sqrt{2}/5 \\ -\sqrt{2}/20 & -3/10 \end{bmatrix}$, $C_1 = \begin{bmatrix} 3/5 & 0 \\ 9\sqrt{2}/20 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & 0 \\ 9\sqrt{2}/20 & -3/10 \end{bmatrix}$, *and* $C_3 = \begin{bmatrix} 0 & 0 \\ -\sqrt{2}/20 & 0 \end{bmatrix}$. Φ *has accuracy* $p = 2$ *and is compactly supported: supp*$(\phi^1) = [0,2]$ *and supp*$(\phi^2) = [0,1]$. *The partition of unity condition (4) holds with* $c_1 = \left(1+\sqrt{2}\right)^{-1}$, $c_2 = \sqrt{2}\left(1+\sqrt{2}\right)^{-1}$. *To satisfy Theorem 4.1 we choose* $\tilde{\phi}^2$ *to be* ϕ^2 *since it is nonnegative. We create* $\tilde{\phi}^1$ *by taking* $\Lambda_1 = \{0,1\}$ *with* $g_{10} = g_{11} = 0.5$*:* $\tilde{\phi}^1(t) = \phi^1(t) + 0.5\left(\phi^2(t) + \phi^2(t-1)\right) \geq 0\ \forall t$. *The new scaling vector* $\tilde{\Phi}$ *partition of unity coefficients from Condition B are* $d_1 = c_1$, $d_2 = c_2 - c_1(g_{10} + g_{11}) > 0$. *Note that* $\tilde{\phi}^1$ *is nonnegative, pictured below in Figure 1. Theorems 4.1 and 4.3 apply to this* $\tilde{\Phi}$. *Notice also that this transformation preserves the symmetry as well as the compact support.*

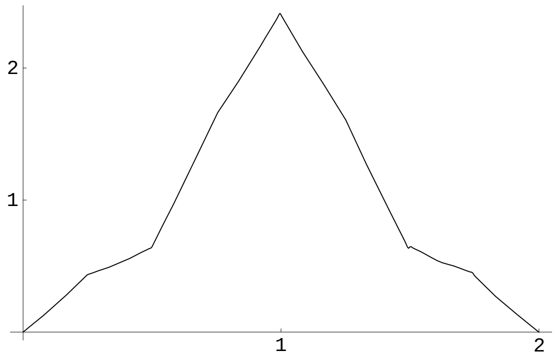

Fig. 1. The positive scaling function $\tilde{\phi}^1$

Example 4.5 *Using a two-scale similarity transform in the frequency domain, Plonka and Strela constructed the following scaling vector* Φ *in [8]. It satisfies the matrix refinement equation* $\Phi(t) = \sum_{k=0}^{2} C_k \Phi(2t-k)$ *where* $C_0 = \frac{1}{20}\begin{bmatrix} -7 & 15 \\ -4 & 10 \end{bmatrix}$, $C_1 = \frac{1}{20}\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}$, *and* $C_2 = \frac{1}{20}\begin{bmatrix} -7 & -15 \\ 4 & 10 \end{bmatrix}$. *This scaling vector is not orthogonal, but it is compactly supported on* $[0,2]$ *with accuracy* $p = 3$. *Moreover,* ϕ^2 *is nonnegative and symmetric about* $t = 1$, *and* ϕ^1 *is antisymmetric about* $t = 1$. *The partition of unity condition (4) holds with* $c_1 = 0$, $c_2 = 1/2$. *To satisfy Theorem 4.1 we choose* $\tilde{\phi}^2$ *to be* ϕ^2 *since it is nonnegative. We create* $\tilde{\phi}^1$ *by taking* $g_{10} = 1.6$*:* $\tilde{\phi}^1(t) = \phi^1(t) + 1.6\phi^2(t) \geq 0$ $\forall t$. *The new scaling vector* $\tilde{\Phi}$ *partition of unity coefficients from Condition B are* $c_1 = d_1 = 0$ *and* $d_2 = c_2 - c_1\left(\sum g_{ik}\right) = c_2 - 0 > 0$. *We observe that creating a nonnegative* $\tilde{\phi}^1$ *was not necessary for avoiding Gibbs', since the kernel*

$K(s,t)$ uses only ϕ^2 and its translates in Theorem 4.3.

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