On Multipower Equations: Some Iterative Solutions and Applications

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Abstract

A generalization of McFarland's iterative scheme [12] for solving quadratic equations in Banach spaces is reported. The notion of a *uniformly contractive system* is introduced and subsequently employed to investigate the convergence of a new iterative method for approximating solutions to this wider class of multipower equations. Existence and uniqueness of solutions are addressed within the framework of a uniformly contractive system. To illustrate the use of the new iterative scheme, we employ it when approximating solutions to a Hammerstein equation and a Chandrashekar equation. Due to the nature of the examples, we have found that wavelet/scaling function bases are a natural choice for the implementation of our iterative method.

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1 Introduction

We seek the solution $x \in X$, where X is a Banach space to the multipower equation:

$$y = x + \lambda L\big(\underbrace{x, \cdots, x}_{k \text{ times}}\big) \tag{1}$$

Here, $y \in X$, and $L : X^k \mapsto X$ is a k-linear operator. The norm of L is given in the usual way:

$$||L|| = \sup_{||x_i|| < 1} ||L(x_1, \ldots, x_k)||.$$

We shall consider for $k \geq 2$ the linear map

$$L_x(\cdot) := L(\underbrace{x, \ldots, x}_{k-1 \text{ times}}, \cdot),$$

and observe that

$$||L_x|| \leq ||L|| \cdot ||x||^{k-1}. \tag{2}$$

The following lemma gives a bound that is useful when measuring the effectiveness of our iterative scheme. The proof of the lemma is similar to that of Theorem 7.3 in [15], and is thus omitted.

Lemma 1.1 *Let $L : X^k \mapsto X$ be a k-linear operator with $k \geq 2$. For $w, z \in X$, $||w||, ||z|| \leq M$,*

$$||L_w - L_z|| \leq (||L|| M^{k-2} (k-1)) ||w - z||. \tag{3}$$

McFarland ([12]) considered $k = 2$ in (1) and derived an iterative scheme for approximating solutions to quadratic equations in Banach spaces. He obtains convergence results for his method by using a continued fraction approach. In order to generalize his method to arbitrary k, we have employed the contractive mapping theorem. We give conditions on the operators involved and the initial guesses that will guarantee convergence of the iterative scheme. The use of the contractive mapping theorem yields a uniqueness of solution result as well.

We then show that these results can be generalized to a wider class of equations where the operator L need not be linear in the first $k-1$ variables, and an applications to boundary value problems is given. We conclude Section 2 with a uniqueness result for "small" solutions of equation (1).

In order to approximate solutions to (1), we consider the fixed point problem

$$Q(x) = x, \tag{4}$$

where, for appropriate $\lambda \in \mathbb{R}$ and $x \in X$, $Q : X \mapsto X$ is given as follows:

$$Q(x) = (\lambda L_x + I)^{-1}(y). \tag{5}$$

The difficulty lies with inverting an infinite dimensional linear operator, so the standard approach is to use successive subspaces $\{V_n\}$ and approximate the solution to the problem in finite dimensional settings. Uniformly contractive systems will be developed to show that these finite dimensional approximations do indeed converge to the true solution of (4).

To formulate the finite dimensional approximating scheme, we first assume that X has a Schauder basis $\{e_k\} \subset X$. Then each $x \in X$ has a unique representation

$$x = \sum_{k=1}^{\infty} <e_k', x> e_k,$$

where $\{e_k'\}_{k \in \mathbb{Z}} \subset X'$ satisfy $<e_k', e_j> = \delta_{kj}$.

Next, let $\{k_n\} \subset \mathbb{N}$ be an increasing sequence. We then define the projection operators $\{S_n\}$ as follows:

$$S_n(x) = \sum_{j=1}^{k_n} <e_j', x> e_j. \tag{6}$$

Since X is complete, $\sup ||S_n|| < \infty$. For convenience, we assume that $||S_n|| = 1$. We then take as our finite dimensional subspaces $V_n = S_n(X)$, and define the linear map $L_x^n : X \mapsto V_n$ as:

$$L_x^n(\cdot) = S_n(L_x(\cdot)) \tag{7}$$

We recall that a k-linear operator $L : X^k \mapsto X$ is *compact* if for any bounded set $B \subset X$, the set $L(B^k)$ is relatively compact. For more details on compact multilinear maps, please see [14, 18, 20].

The next result illustrates that compactness is sufficient to ensure that $L_x^n \to L_x$.

Lemma 1.2 *Suppose that X has a Schauder basis.*

   i) If L is compact in its kth variable, then for each $x \in X$,

$$\lim_{n \to \infty} ||L_x^n - L_x|| = 0 \tag{8}$$

   ii) If L is compact, then

$$\lim_{n \to \infty} ||L^n - L|| = 0 \tag{9}$$

Here, $L^n = S_n L$.

Proof. It is clear that S_n converges uniformly to the identity map I on relatively compact sets. Since L_x is a compact map, for any bounded set $B \subset X$, the set $L_x(B)$ is relatively compact. Now L_x is linear, so

$$||L_x^n - L_x|| = ||(S_n - I)L_x|| = \sup_{||w|| \leq 1} ||(S_n - I)L_x(w)|| \to 0.$$

which proves i).

   If L is compact, then $L(B^k)$ is relatively compact for any bounded set $B \subset X$, so

$$\lim_{n \to \infty} ||L^n - L|| = \lim_{n \to \infty} ||(S_n - I)L|| \to 0. \quad \square$$

   The preceeding result will be used in Section 4 when we derive conditions that guarantee convergence of the sequence of solutions obtained in finite dimensional subspaces to the true fixed point solution of (4). Sufficient conditions on the operator L for this convergence compare favorably with those given in [1].

   In order to obtain the results of Section 4, we have found it convenient to develop the notion of a *uniformly contractive system*. Such a system is defined in Section 3 and is a useful framework with which to show that $z_n \to z$, where z solves (4) and the z_n are the

fixed points of the map $Q_n : X \to V_n$. Uniformly contractive systems have also been used in [19] with another iterative scheme for solving certain nonlinear operator equations.

We conclude the paper with a section of examples illustrating the application of our iterative scheme to approximating solutions of certain multipower equations. We consider approximating solutions to the

i) Hammerstein equation

$$y(s) = x(s) + \lambda \int_a^b K(s,t)(x(t))^2 dt \qquad (10)$$

where $X = L^2[a,b]$.

ii) Chandrashekar equation

$$1 = H(s) + \frac{1}{2}\lambda s H(s) \int_0^1 \frac{H(t)}{s+t} dt \qquad (11)$$

where $X = L^2[0,1]$.

In both cases, we use a sequence of closed nested subspaces $V_0 \subset \cdots \subset V_n \subset X$. Such sequences of subspaces have been found to be particularly useful in many applications when the V_n form a so-called *multiresolution analysis* (see [7], [13], and references therein). We may then employ a sequence of wavelet bases $\{\psi_{nk}\} \subset V_n$ for providing approximate solutions to the multipower equations. These bases are orthonormal and compactly supported. Such properties are desirable in view of the number of integrals that must be computed when we devise a scheme for obtaining approximate solutions to (i) and (ii) above. A discussion of the algorithm used for obtaining approximate solutions is also included in this section.

2 Multipower Operator Equations

We begin this section with a lemma that will be of use later in the section.

Lemma 2.1 *For $k \geq 2$ and $0 < \alpha < \frac{(k-1)^{k-1}}{k^k}$, there exists some $D > 0$ such that*

$$\frac{\alpha(1+D)^{k-1}}{1 - \alpha(1+D)^{k-1}} < D < \frac{1}{k-1} \qquad (12)$$

Proof. Observe that the left inequality in (12) is equivalent to

$$\alpha(1+D)^k - (1+D) + 1 < 0.$$

Now let $f(v) = \alpha v^k - v + 1$. Clearly $f(1) > 0$ and since $\alpha < \frac{(k-1)^{k-1}}{k^k}$, we find that $f(\frac{k}{k-1}) < 0$. Thus there exists some $v = 1 + D$ satisfying $1 < v < \frac{k}{k-1}$ so that the desired inequality (12) holds.$\square$

The following theorem gives conditions on λ, L, and y to ensure that the iteration scheme

$$x_{n+1} = Q(x_n) \tag{13}$$

converges to the true fixed point of (4). Such conditions also lead to the definition of a sphere S wherein any initial guess x_0 will lead to the unique fixed point in sphere $U \subset S$.

Theorem 2.2 *If*

$$0 < |\lambda| \cdot \|L\| \cdot \|y\|^{k-1} = \alpha < \frac{(k-1)^{k-1}}{k^k}, \tag{14}$$

then there exists a solution x_s to (1), unique in the open sphere

$$S = \{z \in X \mid \|z - y\| < \frac{\|y\|}{k-1}\}. \tag{15}$$

If the initial guess $x_0 \in S$, then

$$\lim_{n \to \infty} x_n = x_s \tag{16}$$

where

$$x_{n+1} = Q(x_n).$$

The solution x_s is contained in the closed sphere

$$U = \{z \in X \mid \|z - y\| \leq D\|y\| < \frac{\|y\|}{k-1}\}.$$

where D is given in Lemma 2.1.

Proof. Let $x_0 \in S$ and set $\alpha = |\lambda| \cdot \|L\| \cdot \|y\|^{k-1}$. By Lemma 2.1, we can choose D so that

$$\frac{\|x_0 - y\|}{\|y\|} < D < \frac{1}{k-1}$$

and

$$\frac{\alpha(1 + D)^{k-1}}{1 - \alpha(1 + D)^{k-1}} < D.$$

Note that x_0 is in the closed sphere U. We claim that $Q(U) \subset U$.

To verify our claim, let $x \in U$. Then

$$Q(x) - y = (\lambda L_x + I)^{-1}(I - (\lambda L_x + I))(y)$$

so that

$$
\begin{aligned}
\|Q(x) - y\| &\leq \frac{\|\lambda L_x\|}{1 - \|\lambda L_x\|}\|y\| \\
&\leq \frac{\|\lambda L\|((1 + D)\|y\|)^{k-1}}{1 - \|\lambda L\|((1 + D)\|y\|)^{k-1}}\|y\| \\
&= \frac{\alpha(1 + D)^{k-1}}{1 - \alpha(1 + D)^{k-1}}\|y\| \\
&\leq D\|y\|
\end{aligned}
$$

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Thus $Q(x) \in U$, which proves the claim.

Next we show that $Q : U \mapsto U$ is a contraction mapping, with contraction factor

$$r = \frac{\alpha(1 + D)^{k-2}(k - 1)}{(1 - \alpha(1 + D)^{k-1})^2} < 1$$

(To see that $r < 1$, note that replacing D with $\frac{1}{k-1}$ and α with $\frac{(k-1)^{k-1}}{k^k}$ yields $r = 1$). In order to prove that Q is indeed a contraction, let $x, w \in U$. Then

$$Q(x) - Q(w) = (\lambda L_x + I)^{-1}(\lambda L_w - \lambda L_x)(\lambda L_w + I)^{-1}y$$

which along with Lemma 1.1 and repeated use of (2) yields:

$$
\begin{aligned}
||Q(x) - Q(w)|| &\leq \frac{||\lambda L||(||y||^{k-2}(1 + D)^{k-2}(k - 1))||w - x||}{(1 - ||\lambda L_x||)(1 - ||\lambda L_w||)}||y|| \\
&\leq \frac{\alpha(1 + D)^{k-2}(k - 1)||w - x||}{(1 - ||\lambda L||((1 + D)||y||))^{k-1})^2} \\
&= r||w - x||.
\end{aligned}
\tag{17}
$$

Now since Q is a contraction mapping with $Q(U) \subset U$, we can apply the contraction mapping principle to the iterative scheme $x_{n+1} = Q(x_n)$ and conclude that the iterates must converge to the unique fixed point $x_s \in U$ of Q. Since D can be chosen arbitrarily close to $\frac{1}{k-1}$, the solution must be unique in the open sphere S.$\square$

Corollary 2.3 *The bound $\frac{(k-1)^{k-1}}{k^k}$ on $|\lambda| \cdot ||L|| \cdot ||y||$ in (14) is optimal for $X = \mathbb{R}$.*

Proof. Consider the equation

$$1 = x - \frac{(k - 1)^{k-1}}{k^k}x^k. \quad \square$$

Some remarks are in order before we conclude this section. We first note that the iterative method described in Theorem 2.2 can be generalized slightly to solve equations of the form:

$$y = Ax + \lambda L(x, \cdots x), \tag{18}$$

where A and A^{-1} are linear and bounded. Putting $y^a = A^{-1}y$ and $L^a = A^{-1}L$, we have

$$y^a = x + \lambda L^a(x, \cdots x),$$

which is of the form (1) and can thus be solved using Theorem 2.2.

From Lemma 2.1, we observe that for λ near 0, we can pick D near 0 and conclude that the solution to (1) is close to y, thus improving our choice of the initial guess in the iterative scheme (13).

In the special case $k = 2$, it is useful to compare our results with those obtained elsewhere. In [12], McFarland considered the equation (18) with A invertible and linear. McFarland showed in his Theorem 3 that the iterative scheme

$$x_{n+1} = (A + \lambda L x_n)^{-1} y = (I + \lambda L^a x_n)^{-1} y^a$$

converges to a solution of (18) if

$$0 < ||A^{-1}|| \cdot |\lambda| \cdot ||L|| \cdot ||y^a|| \leq \delta < \frac{1}{4} \tag{19}$$

and if

$$\frac{1 - (1 - 4\delta)^{\frac{1}{2}}}{2} \leq ||\lambda L^a x_0|| < \frac{1}{2}. \tag{20}$$

If our condition (14) with $k = 2$ is satisfied, then so is McFarland's condition (19). Note that McFarland's requirement (20) on the initial condition x_0 may be more difficult to verify than the condition (15) $||x_0 - y|| < ||y||$. McFarland does not use the contraction mapping principle in his proof, and obtains no uniqueness results.

In [17], Rall solves (1) subject to (14), both with $k = 2$. He uses a series approach and shows in his Theorem 19 that the solution x is unique in the sphere

$$\{z \in X \,|\, ||z - x|| < \frac{\sqrt{1 - 4|\lambda|||L||||y||}}{2|\lambda|||L||}\} \tag{21}$$

When $4|\lambda| \cdot ||L|| \cdot ||y||$ is near 1, Rall's uniqueness condition (21) does not give as much information as our condition (15). On the other hand, for λ near 0, Rall's sphere is much larger than our sphere S in (15).

Argyros uses a different iterative method and an auxiliary quadratic equation in [1] to obtain several existence and uniqueness results for equation (1) when $k = 2$. We state a corollary of Argyros [1] for the purpose of comparison to our results.

Corollary 2.4 *(Argyros) For any $y \in X$ such that $||y|| < 4(|\lambda| \cdot ||L||)^{-1}$,*

(i) equation (1) has a unique solution $x \in U(y, a)$, where

$$a = (1 - 2|\lambda| \cdot ||L|| \cdot ||y||)(2|\lambda| \cdot ||L||)^{-1};$$

(ii) moreover $x \in \bar{U}(y, b)$, where

$$b = [1 - 2|\lambda| \cdot ||L|| \cdot ||y|| - (1 - 4|\lambda| \cdot ||L|| \cdot ||y||)^{1/2}](2|\lambda| \cdot ||L||)^{-1}.$$

Argyros' uniqueness ball (i) is bigger than ours (15) but his existence ball (ii) is the same as ours. We summarize this in the following corollary.

Corollary 2.5 *In the case $k = 2$, the solution x_s is in the closed sphere*

$$\begin{aligned} U &= \{z \in X : \|z - y\| \leq \tilde{D}\|y\|\} \\ &= \{z \in X : \|z - y\| \leq (1 - 2|\lambda| \cdot \|L\| \cdot \|y\| - (1 - 4|\lambda| \cdot \|L\| \cdot \|y\|)^{1/2})/(2|\lambda|\|L\|)\}. \end{aligned}$$

Proof. It is easy to see that there exists a unique root $v_0 \in (1, \frac{k}{k-1})$ for the function $f(v)$ given in the proof of Lemma 2.1. Then for each $v \in (v_0, \frac{k}{k-1})$, $D = v - 1$ satisfies inequality (12) and so by Theorem 2.2

$$\|y - x_s\| \leq D\|y\| = (v - 1)\|y\|.$$

Hence $\|y - x_s\| \leq (v_0 - 1)\|y\|$. In the case $k = 2$, the quadratic formula yields

$$(v_0 - 1)\|y\| \leq (1 - 2|\lambda| \cdot \|L\| \cdot \|y\| - (1 - 4|\lambda| \cdot \|L\| \cdot \|y\|)^{1/2})/(2|\lambda|\|L\|).\square$$

The linearity of L in the first $k - 1$ variables is not critical for the results of this section, and we can generalize them somewhat.

Proposition 2.6 *Let $L : X \times X \to X$ satisfy the following:*

(1) $L_x := L(x, \cdot)$ is a bounded linear operator for all $x \in X$,

(2) there exist a $\mu \geq 2$ and $C > 0$ such that

$$\|L(x, v)\| \leq C \|x\|^{\mu - 1} \|v\|$$

and

$$\|L_x - L_v\| \leq C(\mu - 1)(\max(\|x\|, \|v\|))^{\mu - 2} \|x - v\|$$

for all $x, v \in X$.

Then Theorem 2.2 and Corollary 2.5 hold for this operator L with $\|L\|$ and k replaced by C and μ, respectively, throughout the theorem and corollary.

Proof. Note that condition (2) is just a minor generalization of inequalities (2) and (3). The proofs of Theorem 2.2 and Corollary 2.5 are valid with these adjustments. $\square$

It is clear from Lemma 1.1 that a bounded k-linear operator will satisfy the hypotheses of this proposition. The next example is an important example of an operator of this type that is not k-linear.

Example 2.7 *Define $L : C[0, 1] \times C[0, 1] \to C[0, 1]$ by*

$$L(x, v)(s) = \int_0^1 h(s, t) [x(t)]^{\mu - 1} v(t) \, dt,$$

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where $h \in C([0,1]^2)$ and $\mu \geq 2$. Then L is linear in v and

$$\|L(x,v)\|_\infty \leq \|x\|_\infty^{\mu-1} \|v\|_\infty \left\| \int_0^1 h(s,t)\, dt \right\|_\infty.$$

Moreover,

$$
\begin{aligned}
\|(L_x - L_w)(v)\|_\infty &= \left\| \int_0^1 h(s,t)([x(t)]^{\mu-1} - [w(t)]^{\mu-1})v(t)\, dt \right\|_\infty \\
&\leq (\mu-1)M^{\mu-2} \left\| \int_0^1 h(s,t)\, dt \right\|_\infty \|x - w\|_\infty \|v\|_\infty,
\end{aligned}
$$

where $M = \max(\|x\|_\infty, \|w\|_\infty)$, by the lemma below.

Lemma 2.8 *For $x, w \geq 0$, $\mu \geq 2$,*

$$\left| x^{\mu-1} - w^{\mu-1} \right| \leq (\mu-1)(\max(x,\mu))^{\mu-2} |x-w|. \tag{22}$$

Proof. Without loss of generality, assume $w < x$. Fix w and let $f(x) = (\mu-1)x^{\mu-2}(x-w)$, and $g(x) = x^{\mu-1} - w^{\mu-1}$. Then

$$
\begin{aligned}
f'(x) &= (\mu-1)x^{\mu-2} + (x-w)(\mu-1)(\mu-2)x^{\mu-3} \tag{23} \\
&\geq (\mu-1)x^{\mu-2} \\
&= g'(x).
\end{aligned}
$$

Since $f(w) = 0 = g(w)$, inequality (23) yields (22). $\square$

Remark. The class of boundary value problems (see for example [10, 11])

$$x''(t) + \lambda a(t)x^\mu(t) = f(t)$$

with $t \in [0,1]$, $\mu \geq 2$ with appropriate boundary conditions can be transformed into

$$x(t) = \lambda \int_0^1 K(s,t)a(s)x^\mu(s)\,ds + F(t).$$

Then Proposition 2.6 will apply with suitable restrictions on λ and $a(t)$.

For the case $y = 0$ in equation (1), our iterative scheme

$$x_n = (\lambda L_{x_n} + I)^{-1}(y)$$

will yield only the trivial solution $x = 0$. We show below that this is the only "small" solution. To obtain "large" solutions, schemes such as the Newton-Kantorovich method ([2]) can be used. For many problems, the Newton-Kantorovich method will be faster than our iterative scheme. However, if the Fréchet derivative $\lambda L'(x, \ldots, x)$ is not defined or is significantly more expensive to numerically compute than λL_x, then our scheme is preferable.

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Proposition 2.9 *Equation (1) has at most one solution $x \in X$ for which*

$$\|x\| < (\frac{1}{k\,\|L\|})^{\frac{1}{k-1}}. \tag{24}$$

Proof. For the sake of reaching a contradiction, suppose that u and v are distinct solutions to equation (1) satisfying (24). Then

$$
\begin{aligned}
u - v &= L(u, \ldots, u) - L(v, \ldots, v) \\
&= L(u - v, u, \ldots, u) + L(v, u - v, u, \ldots, u) + \cdots + L(v, \ldots, v, u - v)
\end{aligned}
$$

so by (24) we have

$$\|u - v\| \leq k\,\|u - v\|\,\|L\|\,(\max(\|u\|, \|v\|))^{k-1} < \|u - v\|,$$

which is a contradiction. $\square$

3 Uniformly Contractive Systems

We now introduce the notion of a uniformly contractive system (UCS). The role of the UCS is to provide a general framework for obtaining iterative solutions of operator equations that involve contraction mappings. In particular, we will use the concept of the UCS in conjunction with the method discussed in Section 2 to construct approximate solutions to certain multipower equations.

Definition 3.1 *Let X be a Banach space, $\{V_n\}$ a sequence of closed subspaces of X such that*

$$\lim_{n \to \infty} dist(V_n, x) = 0$$

for each $x \in X$. Let U be a closed set in X and define the sets $U_n = V_n \cap U$ and the operators $Q_n : X \mapsto V_n$. We say that $\{U_n, Q_n\}$ is a uniformly contractive system *(UCS) if conditions (C1) and (C2) below hold.*

(C1) There exists a $c \in \mathbb{R}$, $0 < c < 1$, and an $N \in \mathbb{N}$ such that if $n \geq N$ and $x, y \in U$, then

$$Q_n(U) \subset U_n \text{ and } \|Q_n(x) - Q_n(y)\| \leq c\|x - y\|. \tag{25}$$

(C2) For any $x, y \in U$ and $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $k \geq j \geq N$ then

$$\|Q_k(x) - Q_j(y)\| \leq c\|x - y\| + \epsilon. \tag{26}$$

Note that the subspaces $\{V_n\}$ need not be nested, so that the finite element method can be used within the context of a *UCS*.

Theorem 3.2 *Let $\{U_n, Q_n\}$ satisfy (C1) above. Then condition (C2) is equivalent to the existence of a contraction map $Q : U \mapsto U$, defined by $Q(x) = \lim_{n \to \infty} Q_n(x)$, such that*

$$||Q(x) - Q(y)|| \le c||x - y||$$

for $x, y \in U$.

Proof. Assume condition (C2) holds. We first show that the map Q is well defined. Fix $x \in U$ and $\epsilon > 0$. Choose N as prescribed in (C2) and set $y = x$ in (26). Then for $k \ge j \ge N$ we have

$$||Q_k(x) - Q_j(x)|| \le \epsilon.$$

Thus $\{Q_j(x)\}$ is Cauchy. Since X is complete, $\lim_{n \to \infty} Q_n(x)$ exists. Noting that U is closed and $Q_n(x) \in U_n \subset U$ for all n yields

$$\lim_{n \to \infty} Q_n(x) = Q(x) \in U.$$

Now let $x, y \in U$ and $\epsilon > 0$. Choose N so that

$$||Q(x) - Q_N(x)|| < \epsilon \text{ and } ||Q(y) - Q_N(y)|| < \epsilon.$$

Then

$$
\begin{aligned}
||Q(x) - Q(y)|| &\le ||Q(x) - Q_N(x)|| + ||Q_N(x) - Q_N(y)|| + ||Q_N(y) - Q(y)|| \\
&\le \epsilon + c||x - y|| + \epsilon.
\end{aligned}
\tag{27}
$$

Since ϵ is arbitrary, we have

$$||Q(x) - Q(y)|| \le c||x - y||.$$

Next we assume the existence of the map Q, and fix $x, y \in U$ and $\epsilon > 0$. By the definition of Q, there exists an $N \in \mathrm{I\!N}$ such that if $k \ge j \ge N$ then

$$
\begin{aligned}
||Q_k(x) - Q_j(y)|| &\le ||Q_k(x) - Q(x)|| + ||Q(x) - Q(y)|| + ||Q(y) - Q_j(y)|| \\
&\le \epsilon + c||x - y|| + \epsilon
\end{aligned}
\tag{28}
$$

Thus condition (C2) is satisfied. $\square$

We observe that the equations $Q_n(x) = x$ all have unique fixed points $z_n \in U$ by the contraction mapping principle (see [9]). Our next result shows that these fixed points converge to z_s, the unique fixed point of the map $Q \in U$.

Theorem 3.3 *Let $\{U_n, Q_n\}$ be a UCS. Then*

$$\lim_{n \to \infty} z_n = z_s,$$

where

$$Q(z_s) = z_s.$$

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Proof. Let $\epsilon > 0$ and $0 < c < 1$ be the contractive constant for $\{U_n, Q_n\}$. By Theorem 3.2, we can choose $N \in \mathbb{N}$ so that $n \geq N$ implies that

$$||Q(z_s) - Q_n(z_s)|| < (1-c)\epsilon.$$

Then

$$
\begin{aligned}
||z_s - z_n|| &= ||Q(z_s) - Q_n(z_n)|| \\
&\leq ||Q(z_s) - Q_n(z_s)|| + ||Q_n(z_s) - Q_n(z_n)|| \\
&< (1-c)\epsilon + c||z_s - z_n||.
\end{aligned}
$$

So

$$||z_s - z_n||(1-c) < (1-c)\epsilon$$

whence

$$||z_s - z_n|| < \epsilon.$$

Thus $\lim_{n \to \infty} z_n = z_s$. $\square$

Theorem 3.4 *Let $\{U_n, Q_n\}$ be a UCS such that U is bounded and $\{Q_n\}$ converges to Q uniformly on U. Let $N \in \mathbb{N}$ be given as per condition (C1). Beginning with any $k \geq N$ and initial guess $x_k \in U_k$, the iterative scheme*

$$x_{n+k+1} = Q_n(x_{n+k}) \tag{29}$$

will converge to the fixed point of Q in U:

$$\lim_{n \to \infty} x_{n+k} = z_s = Q(z_s).$$

Remark. We note at this time that to numerically implement (29), a hierarchical basis, such as one provided by a multiresolution analysis, is required.

Proof. Fix $\epsilon > 0$. By the uniform convergence of $\{Q_n\}$, there is some $M_1 \geq k \geq N$ such that for $n \geq m \geq M_1$, we have

$$
\begin{aligned}
||x_n - x_m|| &= ||Q_{n-1}(x_{n-1}) - Q_{m-1}(x_{m-1})|| \\
&\leq ||Q_{n-1}(x_{n-1}) - Q_{m-1}(x_{n-1})|| + ||Q_{m-1}(x_{n-1}) - Q_{m-1}(x_{m-1})|| \\
&\leq \epsilon + c||x_{n-1} - x_{m-1}|| \tag{30}
\end{aligned}
$$

Now choose $M_2 > M_1$ such that

$$c^{M_2 - M_1} < \frac{\epsilon}{2\mathrm{diam}(U)}$$

Then for any $n \geq m \geq M_2$, we can repeat the iteration (30) $m - M_1$ times to obtain

$$
\begin{aligned}
||x_n - x_m|| &\leq \epsilon \sum_{j=0}^{m-M_1-1} c^j + c^{m-M_1} ||x_{n-(m-M_1)} - x_{M_1}|| \\
&\leq \epsilon \sum_{j=0}^{m-M_1-1} c^j + 2c^{m-M_1} \operatorname{diam}(U) \\
&\leq \epsilon \left(\sum_{j=0}^{m-M_1-1} c^j + 1 \right)
\end{aligned}
\tag{31}
$$

Thus $\{x_n\}$ is Cauchy, with $\lim_{n \to \infty} x_n = z \in U$.
Now $\{U_n, Q_n\}$ is a UCS, so for $n \geq N$,

$$
\begin{aligned}
||Q(z) - z|| &\leq ||Q(z) - Q_n(z)|| + ||Q_n(z) - Q_n(x_n)|| + ||Q_n(x_n) - z|| \\
&\leq ||Q(z) - Q_n(z)|| + c||z - x_n|| + ||x_{n+1} - z||
\end{aligned}
$$

For n sufficiently large, we have

$$
||Q(z) - z|| \leq ||Q(z) - Q_n(z)|| + c||z - x_n|| + ||x_{n+1} - z|| \leq \epsilon
$$

since $\{Q_n\}$ converges to Q pointwise and $\lim_{n \to \infty} x_n = z$. As ϵ is arbitrary, we have $Q(z) = z$. Since Q has a unique fixed point in U, we conclude that $z_s = z$. $\square$

Remark. The convergence rate for the scheme (29) to the solution z will be governed by the size of the contraction constant c of the UCS, as well as the diameter of U and the uniform convergence of the operators Q_n on U. To be precise, for any given $\epsilon > 0$, there are $M_1, M_2 \in \mathbb{N}$ such that

$$
||Q_n(x) - Q_m(x)|| < \epsilon
$$

for all $m, n \geq M_2 \geq M_1$ and $x \in U$. Then

$$
||z - x_m|| \leq \epsilon \left(\sum_{j=0}^{M_2-M_1} c^j + 1 \right)
$$

for $m \geq M_2$.

4 Applications of Uniformly Contractive Systems for X Possessing a Schauder Basis

Let X be a Banach space with Schauder basis $\{e_k\} \subset X$. Consider the operator

$$
R = (\lambda L_x^n + I) S_n
$$

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as a map from V_n into V_n. Note that $R^{-1} : V_n \mapsto V_n$ exists when $||\lambda L_x^n S_n|| \leq ||\lambda L_x|| < 1$. In matrix terms, with respect to the basis $\{e_k\}$, R^{-1} is formed by inverting the principal submatrix (corresponding to V_n) of the matrix representation of the linear operator $(\lambda L_x^n + I)S_n$.

Let $J_n : V_n \mapsto X$ denote the natural injection operator. Define $(\lambda L_x^n + S_n)^{-1} : X \mapsto X$ by

$$(\lambda L_x^n + S_n)^{-1} = J_n R^{-1} S_n$$

and define $Q_n : X \to V_n$ to be

$$Q_n(x) = (\lambda L_x^n + S_n)^{-1} y \tag{32}$$

We now give convergence conditions for the finite dimensional operators $\{Q_n\}$:

Theorem 4.1 *Suppose that X has a Schauder basis $\{e_k\}$.*

(A) If $L : X^k \mapsto X$ is compact in the kth variable, then Q_n converges to Q pointwise on

$$\{x \in X \ : \ ||\lambda L x|| < 1\}.$$

(B) If $L : X^k \mapsto X$ is compact and $\delta < 1$, then Q_n converges to Q uniformly on

$$U_\delta = \{x \in X \ : \ ||\lambda L|| \cdot ||x||^{k-1} \leq \delta\}.$$

Proof. Let $x \in \{x \in X \ : \ ||\lambda L_x|| < 1\}$ and observe that

$$
\begin{aligned}
Q(x) - Q_n(x) &= (\lambda L_x + I)^{-1} y - (\lambda L_x^n + S_n)^{-1} S_n(y) \\
&= (\lambda L_x + I)^{-1}(y - S_n(y)) + (\lambda L_x + I)^{-1}((\lambda L_x^n + S_n) \\
&\quad - (\lambda L_x + I))(\lambda L_x^n + S_n)^{-1} S_n(y).
\end{aligned}
$$

Since $(\lambda L_x^n + S_n)^{-1} S_n(y) \in V_n$, we have

$$(S_n - I)(\lambda L_x^n + S_n)^{-1} S_n(y) = 0.$$

Thus

$$Q(x) - Q_n(x) = (\lambda L_x + I)^{-1}(y - S_n(y)) + (\lambda L_x + I)^{-1}(\lambda L_x^n - \lambda L_x)(\lambda L_x^n + S_n)^{-1} S_n(y)$$

which yields

$$
\begin{aligned}
||Q(x) - Q_n(x)|| &\leq \frac{1}{1 - ||\lambda L_x||}(||y - S_n(y)|| + ||\lambda L_x^n - \lambda L_x|| \cdot ||(\lambda L_x^n + S_n)^{-1} S_n(y)||) \\
&\leq \frac{||y - S_n(y)||}{1 - ||\lambda L_x||} + \frac{||\lambda L_x^n - \lambda L_x|| \cdot ||y||}{(1 - ||\lambda L_x||)^2}.
\end{aligned}
$$

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To prove (A), we note that for each $x \in X$, L_x is a compact linear map. Now

$$||\lambda L_x^n - \lambda L_x|| \to 0$$

by Lemma 1.2 (i) so that

$$||Q(x) - Q_n(x)|| \to 0$$

thus proving (A).

Next assume that L is compact and that $x \in U_\delta$. Then

$$||Q(x) - Q_n(x)|| \leq \frac{||y - S_n(y)||}{1 - \delta} + \frac{||\lambda L^n - \lambda L|| \cdot ||y||}{(1 - \delta)^2} \cdot \frac{\delta}{||\lambda L||},$$

and since $||\lambda L^n - \lambda L|| \to 0$ uniformly on U_δ by Lemma 1.2 (ii), Q_n converges to Q uniformly on U_δ. This proves (B).□

Remark. For the sake of notation, we have chosen L to be compact in the kth variable. The result holds as long as L is compact in at least one variable.

We now state and prove our main result.

Theorem 4.2 *Suppose that X has a Schauder basis $\{e_k\}$ and that*

$$0 < |\lambda| \cdot ||L|| \cdot ||y||^{k-1} = \alpha < \frac{(k-1)^{k-1}}{k^k}$$

(a) *If L is compact in at least one variable then $\{U_j, Q_j\}$ is a UCS.*

(b) *If L is compact then the iterative scheme (29) given in Theorem 3.4 converges.*

Proof. The first part of the proof is needed for both (a) and (b). Choose $K \in \mathbb{N}$ so that if $j \geq K$ then

$$||S_j(y) - y|| + \frac{\alpha(1 + D)^{k-1}}{1 - \alpha(1 + D)^{k-1}} \leq D$$

where the existence of D is guaranteed by Lemma 2.1. Consider the closed ball

$$U = \{z \in X : ||z - y|| \leq D||y||\}.$$

Then, arguing similarly to the proof of Theorem 2.2, for $x \in U$ and $j \geq K$, we have

$$
\begin{aligned}
||Q_j(x) - y|| &= ||((\lambda L_x^j + S_j)^{-1} - S_j)y + (S_j - I)y|| \\
&= ||(\lambda L_x^j + S_j)^{-1}(S_j - (\lambda L_x^j + S_j))y + (S_j - I)y|| \\
&\leq \frac{||\lambda L_x||}{1 - ||\lambda L_x||}||y|| + ||(S_j - I)y|| \\
&\leq \frac{\alpha(1 + D)^{k-1}}{1 - \alpha(1 + D)^{k-1}}||y|| + ||(S_j - I)y|| \\
&\leq D||y||.
\end{aligned}
$$

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Hence $Q_j(U) \subset U_j$. The proof that each Q_j is a contractive map with the same contraction factor r as Q is very similar to that given for Q in the proof of Theorem 2.2 and is omitted. Thus condition (C1) is satisfied.

To complete the proof of part (a), note that $U \subset \{x \in X : \|\lambda L_x\| < 1\}$ so Theorem 4.1 (A) applies. Then by Theorem 3.2, condition (C2) is satisfied. If L is compact then Q_n converges to Q uniformly on U by Theorem 4.1 (B). Therefore Theorem 3.4 applies.$\square$

We consider the case $k = 2$ so that Theorem 4.2 applies to the quadratic equation

$$y = x + \lambda B(x, x) \tag{33}$$

where B is bilinear. We state Theorem 7 of [1] for comparison purposes.

Theorem 4.3 *(Argyros) Consider the quadratic equations*

$$z = y + F_n(z, z) \tag{34}$$

where $F_n : X \times X \to X$, $n = 1, 2, \ldots$ are bounded symmetric bilinear operators. If

  (i) the sequence $\{F_n\}$ converges to B uniformly as $n \to \infty$,

  (ii) for each n there exists z_n, satisfying (34) and $\sup \|z_n\| < (2|\lambda| \cdot \|B\|)^{-1}$,

then the sequence $\{z_n\}$ converges to a solution z of (33).

Observe that Argyros requires uniform convergence of the operators $\{F_n\}$ to B, while our Theorem 4.2 a) assumes only that B is compact in one variable in order to guarantee that the fixed point solutions z_n converge to the solution z_s of (33).

Also note that Argyros' Theorem requires that the bilinear operators F_n be symmetric, while this is not needed for Theorem 4.2 a). This fact is quite important for an operator $B(x, w)$ that is compact in only one variable, for if B is "symmetrized" using the formula

$$\bar{B}(x, w) := (1/2)(B(x, w) + B(w, x))$$

then the compactness in one variable is destroyed. These points should be kept in mind for Example 5.1 below.

5   Applications to Integral Equations

In the final section of this paper, we employ our iterative scheme and our previous results to obtain approximate solutions to the two classes of integral equations given in the introduction. In both examples, we work in $X = L^2[a, b]$, $-\infty \leq a < b \leq \infty$. While we have considered different finite dimensional subspaces of $L^2(\mathbb{R})$ in our examples, we have found that it is quite beneficial to utilize the closed subspaces $\{V_n\}_{n \in \mathbb{Z}} \subset L^2[a, b]$, $V_n \subset V_{n+1}$, that form a *multiresolution analysis* (MRA) of $L^2[a, b]$ (see [4, 7] for a treatment

of MRA's). The MRA gives rise to a so-called *wavelet basis* $\{\psi_{nk}\}$, where for fixed n, $\{\psi_{nk}\}$ forms an orthonormal basis for the space W_n, $V_{n+1} := V_n \oplus W_n$. For $n, k \in \mathbb{Z}$, $\{\psi_{nk}\}$ forms an orthonormal basis for $L^2[a, b]$. For $L^2(\mathbb{R})$, Daubechies [8] was the first to construct an orthonormal basis $\{\psi_{nk}\}$ generated from one compactly supported *mother wavelet* ψ. For $L^2[a, b]$, $-\infty < a < b < \infty$, similar constructions can be found in [5, 6, 13] among others.

It is desirable to use an orthonormal basis in the subsequent computations since each iterative step involves solving a linear finite dimensional system. In addition, the compact support properties of the wavelet basis greatly reduce the number of numerical integrations that must be performed when we project our operator into finite dimensional subspaces.

Example 5.1 *Consider the* Chandrasekhar *integral equation*

$$1 = H(s) + \frac{1}{2}\lambda s H(s) \int_0^1 \frac{H(t)}{s+t} dt \tag{35}$$

which arises in the theory of radiative transfer [3].

We consider solving (35) in $L^2[0, 1]$ for the function $H(s)$. While this equation is generally solved in $C[0, 1]$, (see [1, 16]), it has been shown in [19] that an L^2 solution to (35) is in fact in $C[0, 1]$. If we put $y(s) = 1$ and define $B : L^2[0, 1] \times L^2[0, 1] \mapsto L^2[0, 1]$ by

$$B(G, H)(s) = \frac{1}{2} s G(s) \int_0^1 \frac{H(t)}{s+t} dt \tag{36}$$

then the Chandrasekhar equation (35) can be expressed in the form

$$y = x + \lambda B(x, x). \tag{37}$$

To obtain a bound on $||B||$, note that

$$
\begin{aligned}
||B(G, H)||^2 &= \frac{1}{4} \int_0^1 (s G(s) \int_0^1 \frac{H(t)}{s+t} dt)^2 ds \\
&\leq \frac{1}{4} \int_0^1 |G(s)|^2 (\int_0^1 (\frac{s}{s+t})^2 dt \int_0^1 (H(t))^2 dt) ds
\end{aligned}
$$

by the Cauchy-Schwarz inequality. Then

$$||B(G, H)||^2 \leq \frac{1}{4} c^2 ||G||^2 ||H||^2$$

where

$$c^2 = \sup_{0 \leq s \leq 1} \int_0^1 (\frac{s}{s+t})^2 dt = \frac{1}{2}$$

so

$$||B|| \leq \frac{1}{2\sqrt{2}}.$$

Note that the linear operator

$$B(G, \cdot)(s) = sG(s) \int_0^1 \frac{\cdot}{s+t} dt$$

is compact for each $G(s) \in L^2[0,1]$. Hence Theorem 4.2 (a) applies for appropriate λ and U, and any orthonormal basis of $L^2[0,1]$. It is worthwhile to note that Theorem 4.1 (B) does not apply, since the operator B is only compact in one variable. It is not clear that Theorem 4.3 of Argyros can be applied to this example since B is only compact in one variable.

This example is an illustration of and not a rationale for Theorem 4.2. Using more information about equation (35) than is used here, it can be proven that (see [3]) (35) has a solution for $0 \leq \lambda \leq 1$, while our results apply for $|\lambda| \leq \frac{1}{\sqrt{2}}$. Our numerical experiments indicate that this iterative scheme converges for $0 \leq \lambda \leq 1$, but we choose a λ value below that is justified by Theorem 4.2(a).

In order to compute approximate solutions to (37), we use the iterative scheme similar to (4):

$$Q_n(x^k) = x^{k+1}, k = 1, 2, \ldots \tag{38}$$

where Q_n was defined by (32), and approximate the fixed point z_n of (38). We then choose larger and larger V_n spaces and repeat the iterative process. Finally, we appeal to Theorem 3.3 to conclude that the z_n approach the true solution.

Suppose we wish to compute the fixed point z_n of (38). Let $\{e_1, \ldots, e_N\}$ be an orthonormal basis for V_n, set $y(t) = 1$, and consider projecting (37) into V_n:

$$(S_n + \lambda B_{x^k}^n) x^{k+1} = y^n$$

where $y^n(t) = \sum_{i=1}^N y_i^n e_i(t)$ with $y_i^n = <y, e_i>$, $i = 1, \ldots, N$. To obtain a matrix representation A^n for $B_{x^k}^n$, we let

$$b_{ij}^k = \frac{1}{2} \int_0^1 \int_0^1 e_k(s) e_i(s) e_j(t) \frac{s}{s+t} dt ds, i, j, k = 1, \ldots, N \tag{39}$$

so that the entries of A^n are given by:

$$a_{pq} = \sum_{i=1}^N b_{iq}^p x_i^k,$$

where

$$x^k(t) = \sum_{i=1}^N x_i^k e_i(t).$$

We start with $x^1 := y$ and then iterate by repeatedly solving the system

$$\begin{bmatrix} y_1^n \\ \vdots \\ y_N^n \end{bmatrix} = (I_n + \lambda A^n) \begin{bmatrix} x_1^{k+1} \\ \vdots \\ x_N^{k+1} \end{bmatrix},$$

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where the $n \times n$ identity matrix I_n is the matrix representation of S_n. Note that the b_{ij}^k's must only be computed once in this scheme and that b_{ij}^k is symmetric in i and k. In addition, if the basis functions have compact support within $[0, 1]$, then it is possible to a priori assign certain $b_{ij}^k = 0$. Certain wavelet bases have this property.

To illustrate the above scheme, we have chosen the first two bases to be $N = 4$ dimensional subspaces while the third choice is an $N = 5$ dimensional subspace. We make no claim as to which is the best choice – we provide these three bases for comparative purposes. Figure 1 below illustrates our approximate solutions.

(i) $P_3([0, 1])$ - the space of polynomials of degree three or less spanned by the classical Legendre polynomials.

(ii) $S_1^2([0, 1])$ - the space of piecewise continuously differentiable quadratic polynomials with possible breakpoints at $0, \frac{1}{2}, 1$, spanned by orthonormalized B-splines. These splines serve as a basis for a $V_1([0, 1])$ space given in Chui and Quak [5].

(iii) $S_0^1([0, 1])$ - the space of continuous linear polynomials with possible breakpoints at $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, spanned by orthonormalized B-splines. These splines serve as a basis for a $V_1([0, 1])$ space given in [5].

Remark. The spaces (ii) and (iii) can be viewed as generated by scaling functions and these functions are compactly supported. Wavelet analysis allows us to readily move to the next larger space using the identity $V_{n+1} = V_n \bigoplus W_n$. As n grows, the support of both the scaling functions and the wavelets shrink thus reducing the number of numerical integrations that need to be computed.

The results of our computations are given in Figure 2 below. In each case, $\lambda = .1$ and six iterations were performed.

The exact solution to (35) is given in [16] and is of the form:

$$H(t) = \exp\left(-\frac{t}{\pi} \int_0^{\pi/2} \frac{\log(1 - \lambda s \cot(s))}{\cos^2(s) + t^2 \sin^2(s)} ds \right) \tag{40}$$

Rall [16] used a different iterative procedure. He computed some values of his approximate solution in a ten-dimensional subspace of $C([0, 1])$ and his computations used seven iterates. For comparison purposes, we list his values with values from our approximate solutions in Table 4 below.

The solution to Chandrashekhar's equation (35) is usually approximated in $C[0, 1]$ rather than $L^2[0, 1]$ (see [17, 1]). Theorem 4.2 is general enough to be applied to $C[0, 1]$ since only a Schauder basis is assumed. A wavelet basis for a dense subspace of $C[0, 1]$ has been reported by Wang [21], so that the benefits of wavelets can be retained if an approximate solution in $C[0, 1]$ is desired.
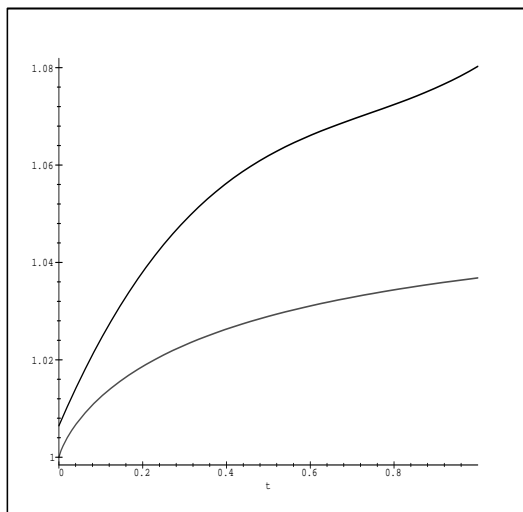
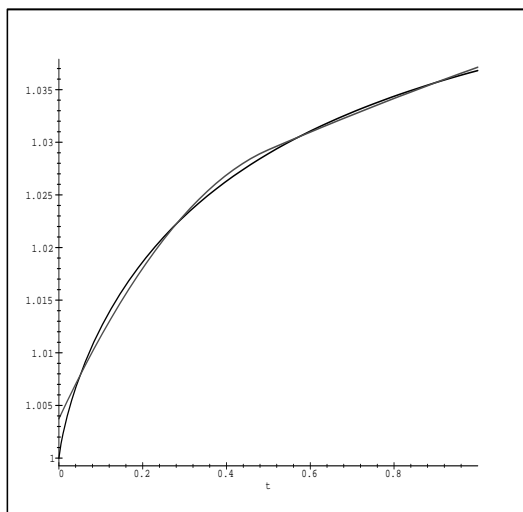**Figure 1:** $H(t)$ (solid) vs. the solution (dotted) from $P_3([0,1])$.

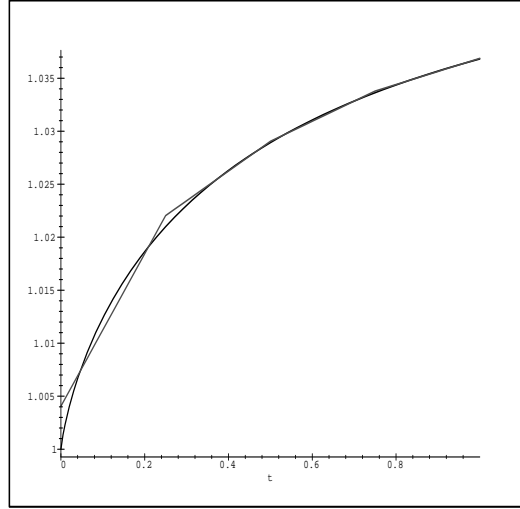**Figure 2:** $H(t)$ (solid) vs. the solution (dotted) $S_1^2([0,1])$.

Figure 3: $H(t)$ (solid) vs. the solution (dotted) $S_0^2([0,1])$.

Table 4.

t	Rall	$P_3([0,1])$	$S_1^2([0,1])$	$S_0^1([0,1])$
0.0159199	1.00333256	1.00958383	1.00506878	1.00519491
0.0819844	1.01089700	1.02134388	1.01025220	1.00994962
0.1933143	1.01829896	1.03723329	1.01764377	1.01796209
0.3378733	1.02435469	1.05173649	1.02472538	1.02450330
0.5000000	1.02892234	1.06190891	1.02928548	1.02904480
0.6621267	1.03220522	1.06818221	1.03198365	1.03211930
0.8066857	1.03445865	1.07263226	1.03426463	1.03448647
0.9180156	1.03589121	1.07655348	1.03594109	1.03586278
0.9840801	1.03664375	1.07947299	1.03690293	1.03667950

Example 5.2 *(Hammerstein) Solve for $x(t) \in L^2[a,b]$:*

$$y(s) = x(s) + \lambda \int_a^b k(s,t)(x(t))^2 dt$$

Equation (5.2) can be expressed as a bilinear equation in $L^2[a,b]$ where

$$K(f_1, f_2)(s) = \int_a^b k(s,t)f_1(t)f_2(t)dt$$

is a bilinear operator,

$$K : L^2[a,b] \times L^2[a,b] \mapsto L^2[a,b].$$

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Using the Cauchy-Schwarz inequality, the map K can be shown to be bounded if

$$k^*(s) = \sup_{a \le t \le b} |k(s,t)| \in L^2[a,b] \tag{41}$$

The linear operator

$$K(f_1, \cdot)(s) = \int_a^b k(s,t) f_1(t)(\cdot) dt$$

is compact for each $f_1(t) \in L^2[a,b]$ when (41) holds. Thus Theorem 4.2 (a) applies. Actually, we can claim that Theorem 4.2 (b) applies, since K is actually compact (see [18]).

We have used the iterative scheme (29). This method is quite similar to that used in Example 5.1 only now each successive iteration takes place in a larger subspace. In the case when $X = L^2(\mathbb{R})$, it is quite convenient to employ the ladder of subspaces given in Daubechies [7].

Example 5.3 *Consider the Hammerstein integral equation*

$$y(s) = x(s) + \lambda \int_a^b k(s, t_1, t_2, ..., t_n) x(t_1) \cdots x(t_n) dt_1 \cdots dt_n \tag{42}$$

As in Example 5.2, (42) can also be viewed as an n-linear equation in $L^2[a,b]$

$$y = x + \lambda L(x, \ldots, x)$$

where the n-linear operator

$$L : L^2[a,b] \times L^2[a,b] \cdots \times L^2[a,b] \mapsto L^2[a,b]$$

is defined by

$$L(f_1, f_2, ..., f_n)(s) = \int_a^b k(s, t_1, t_2, ..., t_n) f_1(t_1) f_2(t_2) ... f_n(t_n) dt_1 dt_2 ... dt_n$$

It can be shown (see [20]) that L is in fact compact when the kernel function is in $L^2([a,b]^{n+1})$, so Theorem 4.2 (b) and the iterative scheme (29) apply.

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