Positive Scaling Vectors on the Interval

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Abstract

In [14], Walter and Shen use an Abel summation technique to construct a positive scaling function P_r, $0 < r < 1$, from an orthonormal scaling function ϕ that generates V_0. A reproducing kernel can in turn be constructed using P_r. This kernel is also positive, has unit integral, and approximations utilitizing it display no Gibbs' phenomenon. These results were extended to scaling vectors and multiwavelets in [12]. In both cases, orthogonality and compact support were lost in the construction process.

In this paper we modify the approach given in [12] to construct compactly supported positive scaling vectors. By imposing certain conditions on the support and the number of components of an existing orthonormal scaling vector Φ, we use ideas from [2,9] to modify Φ and create a positive scaling vector on the interval.

While the mapping into V_0 associated with this new positive scaling vector is not a projection, the scaling vector does produce a Riesz basis for V_0 and we conclude the paper by illustrating that expansions of functions via positive scaling vectors exhibit no Gibbs' phenomenon.

Key words: scaling functions, scaling vectors, Gibbs' phenomenon, summability techniques, compactly supported scaling vectors

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1 Introduction

Let ϕ be a compacted supported orthogonal scaling function generating a multiresolution analysis $\{V_k\}$ for $L^2(\mathbb{R})$. In [14], the authors show how to use this ϕ to construct a new scaling function P that generates the same multiresolution analysis for $L^2(\mathbb{R})$. Moreover, $P(t) \geq 0$ for $t \in \mathbb{R}$. The application the authors considered for this new function P was density estimation. They also showed that approximations $f_m \in V_m$ to $f \in L^2(\mathbb{R})$ where

$$f_m(t) = \int\limits_{s\in\mathbb{R}} K_m(s,t)f(s)ds$$

where

$$K_m(s,t) = 2^m \sum_{n\in\mathbb{Z}} \phi(2^m s - n)\phi(2^m t - n)$$

exhibits no Gibbs' phenomenon. While K_m is not a projection of f into V_m, f_m may well be useful in some applications where Gibbs' phenomenon is a problem. The disadvantages of this construction is that orthogonality is lost (although the authors gave a simple expression for the dual P^*) and P is not compactly supported.

The results of Walter and Shen [14] were generalized to the scaling vectors $\Phi = (\phi^1, \ldots, \phi^A)^T$ in [12]. Here the authors also showed that it was not necessary to start with an orthogonal scaling vector supported on some interval $[0, M]$ to construct the nonnegative scaling vector P.

While the orthogonality of a scaling vector is desirable in some cases, it is impossible to insist that the scaling vector of length $A > 1$ be both orthogonal and nonnegative. As we will see in the sequel is it possible to modify the construction and retain the compact support. Moreover, since many applications require the underlying space to be $L^2([a,b])$ rather than $L^2(\mathbb{R})$ it is worthwhile to investigate extending the construction to the interval.

In this paper, we will take a continuous, compactly supported scaling vector Φ and illustrate how to construct a compactly supported scaling vector P that generates the same multiresolution analysis as Φ. The construction only requires that at least one component ϕ^j of Φ is nonnegative on its support. We then show how to construct the new scaling vector P by taking $p^j = \phi^j$ and forming p^ℓ from ϕ^ℓ and a linear combination of the integer translates of ϕ^j.

We continue by discussing a method for modifying a given scaling vector Φ and constructing a scaling vector P for the interval. This construction is motivated by the work of Daubechies' [2] and Meyer [9]. It is a goal of the construction to produce a positive scaling vector, preserve the polynomial accuracy of the original scaling vector, and to keep the number of edge functions as small as possible. Our results are partial in that we only consider $A = 2$ and vectors that satisfy certain support properties. With regards to the number of edge functions in the construction, we found that only $m-1$ edge functions were needed to preserve polynomial accuracy m.

We consider also the question of Gibbs' phenomenon for scaling vector expansions. The main result of this portion of this paper is to show that if Φ is orthogonal or Φ has a biorthogonal dual that is compactly supported, then the corresponding wavelet expansion exhibits Gibbs' phenomenon on at least one side of 0.

The outline of this paper is as follows. In the next section, we introduce basic definitions, examples, and results that are used throughout the sequel. In Section 3, we give the construction for a scaling vector that is compactly supported. In the subsequent section we show how to take a given scaling vector and use it to construct a scaling vector that generates a muliresolution analysis for $L^2([0,1])$. We give examples of the constructions detailed in Section 3 and Section 4. The final section contains our results involving wavelet expansions and Gibbs' phenomenon.

2 Notation, Definitions, and Prelimary Results

In this section we will state definitions, introduce notation, and present results used throughout the sequel.

We begin with the concept of a scaling vector or set of multiscaling functions. This idea was first introduced in [4,7]. We start with A functions, $\phi^1, \ldots, \phi^A$ and consider the space

$$V_0 = \overline{< \{\phi^1(\cdot - k), \ldots, \phi^A(\cdot - k)\}_{k \in \mathbb{Z}} >}.$$

It is convenient to store $\phi^1, \ldots, \phi^A$ in a vector

$$\Phi(t) = \begin{pmatrix} \phi^1(t) \\ \phi^2(t) \\ \vdots \\ \phi^A(t) \end{pmatrix}$$

and define a *multiresolution analysis* in much the same manner as in [2]:

(M1) $\overline{\cup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$.
(M2) $\cap_{n\in\mathbb{Z}} V_n = \{0\}$.
(M3)$f \in V_n \leftrightarrow f(2^{-n}\cdot) \in V_0$, $n \in \mathbb{Z}$.
(M4) $f \in V_0 \rightarrow f(\cdot - n) \in V_0$, $n \in \mathbb{Z}$.
(M5) Φ generates a Reisz basis for V_0.

In this case Φ satifies a *matrix refinement equation*:

$$\Phi(t) = \sum_k C_k \Phi(2t - k) \tag{1}$$

where the C_k are $A \times A$ matrices.

We define the *Fourier transform* $\hat{\Phi}$ of Φ by the component-wise rule:

$$\hat{\phi}^\ell(\omega) = \int_{\mathbb{R}} \phi^\ell(t) e^{-i\omega t} dt,\ \ \ell = 1, \ldots, A$$

and the $A \times A$ matrix

$$E_\Phi(\omega) = \sum_{k\in\mathbb{Z}} \hat{\Phi}(\omega + 2\pi k)\hat{\Phi}^\dagger(\omega + 2\pi k) \tag{2}$$

where $\dagger$ denotes the Hermitian conjugate. The matrix E_Φ plays an important role in analyzing scaling vectors. Indeed Geronimo, Hardin, and Massopust introduced this matrix in [4] and showed that the nonsingularity of E_Φ is necessary and sufficient for the set in (M5) to form a Riesz basis for V_0.

We will make the following assumptions about Φ and its components:

(A1) Each ϕ^ℓ is compactly supported and continuous,
(A2) There is a vector $\vec{c} = (c_1, \ldots, c_A)^T$ for which

$$\sum_{\ell=1}^{A} \sum_{k\in\mathbb{Z}} c_\ell \phi^\ell(t - k) = 1,$$

(A3) $\sum_{k\in\mathbb{Z}} \phi^\ell(t-k) > 0$ for each $\ell \in \mathbb{Z}$ such that $c_\ell \neq 0$, and
(A4) $c_\ell \int_{\mathbb{R}} \phi^\ell \geq 0$ for each $\ell = 1, \ldots, A$ and if $\int_{\mathbb{R}} \phi^\ell = 0$, then $c_\ell = 0$.

It was shown by Theorem 3.1 in [12] that if Φ is a scaling vector satisfying (M1)-(M5) and (A1)-(A2), a new scaling vector $\tilde{\Phi}$ could be constructed that generates the same multiresolution analysis as Φ and also satisfies (A3)-(A4).

With regards to (A1), we will further assume that

$$\mathrm{supp}(\phi^\ell) = [0, M_\ell]$$

with $M_\ell \in \mathbb{Z}$ for $\ell = 1, \ldots, A$.

We will denote by M the maximum value of M_ℓ:

$$M = \mathrm{Max}\{M_1, \ldots, M_A\}. \tag{3}$$

We will say that Φ has *polynomial accuracy* m if there exist constants f^ℓ_{nk} such that

$$t^n = \sum_{\ell=1}^{A} \sum_{k\in\mathbb{Z}} f^\ell_{nk}\phi^\ell(t-k) = \sum_{k\in\mathbb{Z}} \mathbf{f}_{nk} \cdot \Phi(t-k) \tag{4}$$

for $n = 0, \ldots, m-1$ and

$$\mathbf{f}_{nk} = (f^1_{nk}, \ldots, f^A_{nk})^T \tag{5}$$

We now give two examples of multiscaling functions that we will use throughout the sequel.

Example 2.1 (Donovan,Geronimo,Hardin,Massopust) *In [3], the authors constructed a scaling vector with $A = 2$ that satisfies the four-term matrix refinement equation*

$$\Phi(t) = \sum_{k=0}^{3} C_k \Phi(2t-k)$$

where $C_0 = \begin{bmatrix} 3/5 & 4\sqrt{2}/5 \\ -\sqrt{2}/20 & -3/10 \end{bmatrix}$, $C_1 = \begin{bmatrix} 3/5 & 0 \\ 9\sqrt{2}/20 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & 0 \\ 9\sqrt{2}/20 & -3/10 \end{bmatrix}$, *and* $C_3 = \begin{bmatrix} 0 & 0 \\ -\sqrt{2}/20 & 0 \end{bmatrix}$.

The scaling functions ϕ^1, ϕ^2 (shown in Figure 1) satisfy $\left(\phi^j(t-n), \phi^\ell(t-m)\right) = \delta_{mn}\delta_{j\ell}$, $m, n \in \mathbb{Z}$, $j, \ell = 1, 2$. They also have approximation order 2, are continuous, symmetric, and compactly supported (supp$(\phi^1) = [0, 2]$ and supp$(\phi^2) = [0, 1]$). Note also that $\phi^2(t) \geq 0$, $t \in \mathbb{R}$.

Our next example uses a scaling vector constructed by Plonka and Strela in [11].

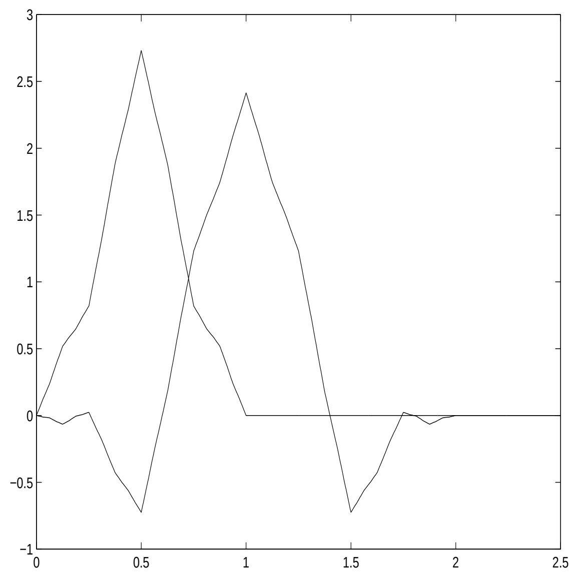

Fig. 1. The scaling functions ϕ^1, ϕ^2 of Donovan, Geronimo, Hardin, and Massopust.

Example 2.2 (Plonka,Strela) *Using a two-scale similarity transform in the frequency domain, Plonka and Strela constructed the following scaling vector* Φ *in [11]. It satisfies a three-term matrix refinement equation*

$$\Phi(t) = \sum_{k=0}^{2} C_k \Phi(2t-k)$$

where $C_0 = \frac{1}{20}\begin{bmatrix} -7 & 15 \\ -4 & 10 \end{bmatrix}$, $C_1 = \frac{1}{20}\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}$, *and* $C_2 = \frac{1}{20}\begin{bmatrix} -7 & -15 \\ 4 & 10 \end{bmatrix}$.

This scaling vector is not orthogonal, but it is compactly supported on $[0,2]$ *and (anti)symmetric with approximation order 3. Moreover,* ϕ^2 *is nonnegative on its support.*

3 Positive Scaling Vectors with Compact Support

In this section we describe a procedure for constructing positive scaling vectors with compact support. The idea is to start with a scaling vector Φ that satisfies (A1)-(A2) from the previous section and with the addition requirement that at least one of components ϕ^j of Φ is nonnegative.

Theorem 3.1 *Suppose a scaling vector* $\Phi = (\phi^1, \phi^2, \ldots, \phi^A)^T$ *satisfies (A1)-(A2) and for some* $j = 1, \ldots, A$, $\phi^j(x) \geq 0$ *for all* $x \in \mathbb{R}$. *Suppose there exist*

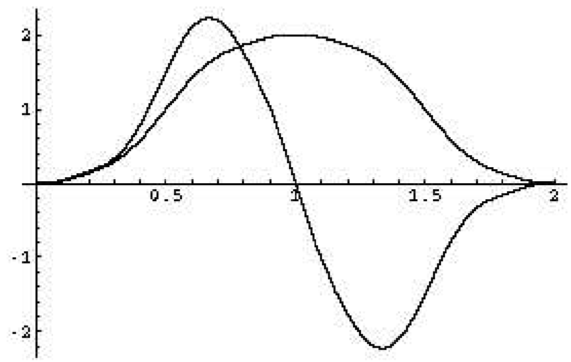

Fig. 2. The Φ of Plonka and Strela.

finite index sets Λ_i and constants c_{ik} for $i \neq j$ such that the function

$$\tilde{\phi}^i(t) := \phi^i(t) + \sum_{k \in \Lambda_i} c_{ik} \phi^j(t-k) \geq 0 \tag{6}$$

for all $x \in \mathbb{R}$. Then the nonnegative vector

$$\tilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^{j-1}, \phi^j, \tilde{\phi}^{j+1}, \ldots, \tilde{\phi}^A)^T$$

is a scaling vector that satisfies (A1)-(A4) and generates the same space V_0 as Φ.

Proof. We must show that $\tilde{\Phi}$ satisfies a matrix refinement equation in addition to (A1)-(A2). By Theorem 3.1 in [12], we can then conclude $\tilde{\Phi}$ also satisfies (A3)-(A4).

(A1) follows immediately from the support and continuity properties of Φ. To prove (A2) we start by rewriting (6) as

$$\phi^i(t) = \tilde{\phi}^i(t) - \sum_{k \in \Lambda_i} c_{ik} \phi^j(t-k)$$

and substitute this into the original partition of unity:

$$\sum_{n \in \mathbb{Z}} \left\{ \sum_{i \neq j} c_i \left[\tilde{\phi}^i(t-n) - \sum_{k \in \Lambda_i} c_{ik} \phi^j(t-k-n) \right] + c_j \phi^j(t-n) \right\} = 1$$

so that

$$\sum_{n\in\mathbb{Z}}\sum_{i\neq j} c_i\tilde{\phi}^i(t-n) - \sum_{i\neq j}\sum_{k\in\Lambda_i} c_i c_{ik}\sum_{n\in\mathbb{Z}}\phi^j(t-k-n) + \sum_{n\in\mathbb{Z}} c_j\phi^j(t-n) = 1.$$

Substituting $m = n + k$ into the second expression gives:

$$\sum_{n\in\mathbb{Z}}\sum_{i\neq j} c_i\tilde{\phi}^i(t-n) - \sum_{i\neq j}\sum_{k\in\Lambda_i} c_i c_{ik}\sum_{m\in\mathbb{Z}}\phi^j(t-m) + \sum_{n\in\mathbb{Z}} c_j\phi^j(t-n) = 1$$

or

$$\sum_{n\in\mathbb{Z}}\sum_{i\neq j} c_i\tilde{\phi}^i(t-n) + \sum_{n\in\mathbb{Z}}\left\{c_j - \sum_{i\neq j}\sum_{k\in\Lambda_i} c_i c_{ik}\right\}\phi^j(t-n) = 1.$$

Since $\tilde{\phi}^j = \phi^j$, we get the partition of unity

$$\sum_{i=1}^{A}\sum_{n\in\mathbb{Z}} d_i\tilde{\phi}^i(t-n) = 1$$

where $d_i = c_i$ for $i \neq j$ and

$$d_j = c_j - \sum_{i\neq j}\sum_{k\in\Lambda_i} c_i c_{ik}.$$

To see that $\tilde{\Phi}$ forms a Riesz basis for V_0, assume without loss of generality that $j = A$ and note that

$$\hat{\tilde{\Phi}}(\omega) = B(\omega)\hat{\Phi}(\omega)$$

where $B(\omega)$ is an $A \times A$ upper triangular matrix defined by

$$B(\omega) = \begin{bmatrix} I_{A-1} & \vec{m} \\ \vec{0} & 1 \end{bmatrix}$$

where I_{A-1} is the $A-1 \times A-1$ identity matrix, $\vec{0}$ is a $A-1$ row vector of 0's, and $\vec{m}$ is an $A-1$ column vector whose components m_i, $i = 1, \ldots, A-1$, are given by

$$m_i = \sum_{k\in\Lambda_i} c_{ik}e^{-ik\omega},$$

We compute the $A \times A$ matrix

$$\begin{aligned}
E_{\tilde{\Phi}}(\omega) &= \sum_{k \in \mathbb{Z}} \hat{\tilde{\Phi}}(\omega + 2\pi k)\hat{\tilde{\Phi}}^{\dagger}(\omega + 2\pi k) \\
&= \sum_{k \in \mathbb{Z}} B(\omega + 2\pi k)\hat{\Phi}(\omega + 2\pi k)\hat{\Phi}^{\dagger}(\omega + 2\pi k)B^{\dagger}(\omega + 2\pi k) \\
&= B(\omega)\left(\sum_{k \in \mathbb{Z}} \hat{\Phi}(\omega + 2\pi k)\hat{\Phi}^{\dagger}(\omega + 2\pi k)\right)B^{\dagger}(\omega) \\
&= B(\omega)E_{\Phi}(\omega)B^{\dagger}(\omega)
\end{aligned}$$

By definition $B(\omega)$ is nonsingular, so that $B^{\dagger}(\omega)$ is also nonsingular. Since Φ forms a Riesz basis for V_0, we have that $E_{\Phi}(\omega)$ is also nonsingular. Thus $E_{\tilde{\Phi}}(\omega)$ is nonsingular and thus by virtue of Theorem 3.2 in [4], $\tilde{\Phi}$ generates a Riesz basis for V_0.

We must finally show that $\tilde{\Phi}$ satisfies a matrix refinement equation. Let

$$B = \begin{bmatrix} I_{A-1} & \vec{c} \\ \vec{0} & 1 \end{bmatrix}$$

where I_{A-1} and $\vec{0}$ are as defined above and the components c_i, $i = 1, \ldots, A-1$ of $\vec{c}$ are given by

$$c_i = \sum_{k \in \Lambda_i} c_{ik}.$$

Then

$$\begin{aligned}
\tilde{\Phi}(t) &= B\Phi(t) \\
&= B\sum_{k} C_k \Phi(2t - k) \\
&= \sum_{k} BC_k \Phi(2t - k).
\end{aligned}$$

But B is nonsingular so that we can write

$$\Phi(2t - k) = B^{-1}\tilde{\Phi}(2t - k)$$

and thus observe that the refine equation coefficients for $\tilde{\Phi}$ are

$$\tilde{C}_k = BC_kB^{-1} \; \square.$$

Remark. A sufficient condition on ϕ^j for the existence of these index sets is $\phi^j > 0$ on an interval J, where $\bar{J} = [a, b]$ and $b - a > 1$.

We conclude by giving two examples that illustrate the results of Theorem 3.1. The first example involves the scaling vector of Donovan, Geronimo, Hardin, and Massopust [3] while the second utilizes the vector constructed by Plonka and Strela [11].

Example 3.2 *We consider the multiscaling functions given in Example 2.1. Since $\phi^2 \geq 0$ we take it to be $\tilde{\phi}^2$. We can then create $\tilde{\phi}^1$ by taking $\Lambda_1 = \{0, 1\}$ with $c_{1k} = 1$, $k = 0, 1$:*

$$\tilde{\phi}^1(t) = \phi^1(t) + \big(\phi^2(t) + \phi^2(t-1)\big)$$

$\tilde{\phi}^1(t)$ is pictured below in Figure 3:

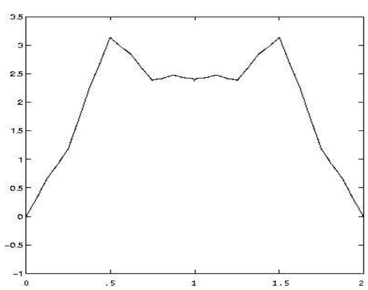

Fig. 3. The positive scaling function $\tilde{\phi}^1$.

Our next example uses a scaling vector constructed by Plonka and Strela described in Example 2.2.

Example 3.3 *We take $\tilde{\phi}^2(t) = \phi^2(t)$ and take $\alpha = 1.6$. Then $\alpha > |\min_{[0,2]}(\phi^1(t) + \phi^2(t))|$. We then form*

$$\tilde{\phi}^1(t) = \phi^1(t) + \alpha\phi^2(t)$$

so that $\tilde{\phi}^1(t) \geq 0$ for $t \in \mathbb{R}$. $\tilde{\phi}^1(t)$ is pictured below.

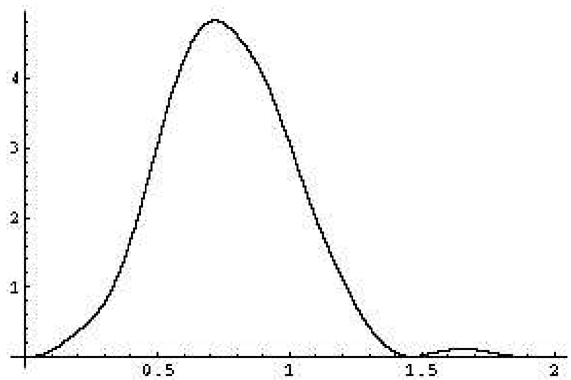

Fig. 4. $\tilde{\phi}^1$.

4 Positive Scaling Vectors on $[0, 1]$

The construction of scaling vectors on the interval has been addressed in [1], [6], and [8]. In these cases the authors constructed scaling vectors on the interval from scratch. It is our intent to show how to modify a given scaling vector that generates a multiresolution analysis for $L^2(\mathbb{R})$ so that it generates a multiresolution analysis for $L^2([0, 1])$. Moreover, the components of the new vector will be nonnegative. The disadvantages to this scheme are that orthogonality is lost and that the new scaling vector is not a projection into V_0. The advantage is that multiwavelet expansions of f in V_m exhibit no Gibbs' phenomenon. In particular cases, our procedure requires fewer edge functions than in the single scaling function constructions of Daubechies [2] and Meyer [9].

Our task then is to modify an existing scaling vector and create a nonnegative scaling vector that generates a multiresolution analysis for $L^2([0, 1])$ that

(1) preserves the polynomial accuracy of the original scaling vector.
(2) exhibits no Gibbs' phenomemon
(3) avoids the creation of "too many" edge functions.

Unfortunately we only have results in particular cases. It is our hope that these results might be extended to more general cases in the future.

We begin with a multiresolution analysis for $L^2(\mathbb{R})$ generated by $\Phi = (\phi^1, \phi^2, \ldots, \phi^A)^T$ and we will also assume our scaling vector has polynomial accuracy m with $\mathbf{f}_{nk}$ given by (5).

Finally, assume that the set S of non-zero functions

$$S = \{\bar{\phi}^{\ell}(\cdot - k), k \in \mathbb{Z}\} \tag{7}$$

defined by

$$\bar{\phi}^{\ell}_k(t) = \phi^{\ell}(t-k)|_{[0,\infty)}$$

are linearly independent and let $n(S)$ denote the number of functions in S.

We will work only on the left edge $[0,\infty)$ in constructing $V_0[0,\infty)$. We begin with $\phi^{\ell}(t-k)$, $k \leq 0$ and then add left edge functions to preserve polynomial accuracy.

Define the left edge functions $\phi_{L,n}$ by

$$\phi_{L,n}(t) = \sum_{\ell=1}^{A} \sum_{k=1-M_\ell}^{0} f^{\ell}_{nk} \phi^{\ell}(t-k)|_{[0,\infty)} = \sum_{\ell=1}^{A} \sum_{k=1-M_\ell}^{0} f^{\ell}_{nk} \bar{\phi}^{\ell}_k(t) \tag{8}$$

for $n = 0, \dots, m-1$. Observe that the sum above is finite since the ϕ^{ℓ} are compactly supported and note that by (4) $\phi_{L,n}(t) = t^n$ on $[0,1]$. Right edge functions can be defined in an analogous manner.

Our next proposition shows that the left edge functions (and in an analogous manner the right edge functions) satisfy a matrix refinement equation.

Proposition 4.1 *Suppose that Φ is a scaling vector satisfying (A1)-(A2) with polynomial accuracy m with f^{ℓ}_{nk} given in (4). Further assume that the set S defined in (7) is linearly independent. Then the set of edge functions $\phi_{L,n}$, $n = 0, \dots, m-1$ satisfy a matrix refinement equation.*

Proof. Recall Φ supported on $[0, M]$ satisfies a matrix refinement equation

$$\Phi(t) = \sum_{j=0}^{M} C_j \Phi(2t - j)$$

or

$$\begin{aligned}
\Phi(t-k) &= \sum_{j=0}^{M} C_j \Phi(2(t-k) - j) \\
&= \sum_{j=2k}^{2k+M} C_{j-2k} \Phi(2t - j) \\
&= \sum_{j \in \mathbb{Z}} C_{j-2k} \Phi(2t - j)
\end{aligned}$$

where $C_t = 0$ if $t \notin \{0, \ldots, M-1\}$.

Note that for each $n = 0, \ldots, m-1$,

$$\begin{aligned}
\phi_{L,n}(t) - 2^{-n}\phi_{L,n}(2t) &= \sum_{k=1-M}^{0} \mathbf{f}_{nk}\Phi(t-k) - 2^{-n}\sum_{k=1-M}^{0} \mathbf{f}_{nk}\Phi(2t-k) \\
&= \sum_{k=1-M}^{0} \mathbf{f}_{nk}\sum_{j\in\mathbb{Z}} C_{j-2k}\Phi(2t-j) - 2^{-n}\sum_{k=1-M}^{0} \mathbf{f}_{nk}\Phi(2t-k) \\
&= \sum_{j\in\mathbb{Z}}\sum_{k=1-M}^{0} \mathbf{f}_{nk}C_{j-2k}\Phi(2t-j) - 2^{-n}\sum_{k=1-M}^{0} \mathbf{f}_{nk}\Phi(2t-k) \\
&= \sum_{j=2-2M}^{M} \mathbf{q}_{nj}\Phi(2t-j)
\end{aligned}$$

on $[0,\infty)$ where

$$\mathbf{q}_{nj} = \begin{cases} \sum_{k=1-M}^{0} \mathbf{f}_{nk}C_{j-2k} - 2^{-n}\mathbf{f}_{nj} & \text{if} j \in \{1-M, \ldots, 0\} \\ \sum_{k=1-M}^{0} \mathbf{f}_{nk}C_{j-2k} & \text{if} j \in \{2-2M, \ldots, -M\} \cup \{1, \ldots, M\} \end{cases}$$

Recall that $\phi_{L,n}(t) = 2^{-n}\phi_{L,n}(2t) = t^n$ on $[0, \frac{1}{2}]$ and that the functions $\phi^\ell(2t-j)$ are linearly independent so

$$\mathbf{q}_{nj} = 0$$

for $j = 2-2M, \ldots, 0$. Thus

$$\phi_{L,n}(t) = 2^{-n}\phi_{L,n}(2t) + \sum_{j=1}^{M} \mathbf{q}_{nj}\Phi(2t-j)$$

on $[0,\infty)$. This is the desired dilation equation for the n^{th} edge function $\phi_{L,n}$. □

Refinement equations for the right edge functions are derived in a similar manner.

Example 4.2 *We return to the scaling vector of Strela and Plonka [11] introduced in Example 3.3. This scaling vector has approximation order 3 with*

$$\mathbf{f}_{0,0} = (0,1), \qquad \mathbf{f}_{1,0} = (-\frac{1}{6}, 1).$$

and is supported on $[0,2]$. *The refinement equation matrices* C_0, C_1, *and* C_2

are given in Example 3.3. We calculate $\mathbf{q}_{0,1}$ *and* $\mathbf{q}_{0,2}$ *as*

$$\mathbf{q}_{0,1} = \sum_{k=1-2}^{0} \mathbf{f}_{0,k} C_{1-2k} = (0,1)C_1 = (0,1)$$

$$\mathbf{q}_{0,2} = \sum_{k=1-2}^{0} \mathbf{f}_{0,k} C_{2-2k} = (0,1)C_2 = (\frac{1}{5}, \frac{1}{2}).$$

The dilation equation for $\phi_{L,0}$ *is*

$$\phi_{L,0}(t) = \phi_{L,0}(2t) + \phi^2(2t-1) + \frac{1}{5}\phi^1(2t-2) + \frac{1}{2}\phi^2(2t-2)$$

and the dilation equation for $\phi_{L,1}$ *is*

$$\phi_{L,1}(t) = 2^{-1}\phi_{L,1}(2t) - \frac{1}{12}\phi^1(2t-1) + \phi^2(2t-1) + \frac{31}{120}\phi^1(2t-2) + \frac{75}{120}\phi^2(2t-2).$$

In order to construct a scaling vector for $V_0([0,\infty))$, we need for our edge functions not only to satisfy a matrix refinement equation but also to join with $\{\Phi(\cdot - k)\}_{k\geq 0}$ and form a Riesz basis for $V_0([0,\infty])$. We will next show that the set of edge functions we constructed above does indeed preserve the Riesz basis property. We begin with the following lemma.

Lemma 4.3 *Let* $\{e_k\}$ *be a Riesz basis of some space* H*. If* $T : H \to K$ *is linear and invertible, with* T *and* T^{-1} *both continuous, then* $\{Te_k\}$ *is also a Riesz basis of* K*. In particular, if* $H \subset K$*,* T *is injective and* $T(e_k) = e_k$ *for all but a finite number of the* e_k*, then* T *and* T^{-1} *are continuous.*

We are now ready to state and prove our next result.

Theorem 4.4 *Let* $\Phi = (\phi^1, \ldots, \phi^A)^T$ *be a scaling vector that satisfies (A1) and generates a multiresolution analysis for* $L^2(\mathbb{R})$*. For some index set* B*, let* $\{L_i\}_{i\in B}$ *be a finite set of edge functions with* $supp(L_i) = [0, \delta_i]$ *and assume that* $\{L_i, \phi^\ell(\cdot - k)\}_{i,\ell,k\geq 0}$ *is a linearly independent set. Then* $\{L_i(2^j\cdot), \phi^\ell(2^j \cdot -k)\}_{i,\ell,k\geq 0}$ *is a Riesz basis of* V_j*, where* $L^2([0,\infty)) = \overline{\cup_j V_j}$*.*

Proof. Without loss of generality set $j = 0$ and let C be the set of integer indices for which $\text{supp}(L_i) \cap \text{supp}(\phi^\ell(\cdot - k)) \neq \emptyset$ for all $i \in B, \ell, k \geq 0$. For ease of notation, denote by $\{f_n\}_{n\in C}$ those $\{\phi^\ell(\cdot - k)\}$ corresponding to C and for integer index set D let $\{g_m\}_{m\in D}$ denote the other $\{\phi^\ell(\cdot - k)\}$. For ease of presentation, assume that B,C, and D are mutually disjoint and note that $\mathbb{Z} = B \cup C \cup D$. Now since $\{L_i, f_n\}$ is linearly independent and a finite set, it must be a Riesz basis of its span. We then use the Gram-Schmidt process to orthogonalize it and thus obtain $\{\tilde{L}_i, \tilde{f}_n\}$. In the process, we begin with the L_i and then move on to the $\{f_n\}$. This ensures that $\mathrm{supp}(\tilde{L}_i) \subset [0, \max(\delta_j)]$, whence $\int \tilde{L}_i g_n = 0$ for all i, n.

Since there are only a finite number of f_n functions, $\{\tilde{f}_n, g_m\}$ is still a Riesz basis of its span by the Lemma 4.3. Hence there exist $A_0, B_0 > 0$ such that

$$A_0 ||\{d_k\}||_2^2 \leq || \sum_{n \in C} d_n \tilde{f}_n + \sum_{m \in D} d_m g_m ||_2^2 \leq B_0 ||\{d_k\}||_2^2, \qquad \forall \{d_k\} \in \ell^2(\mathbb{Z}).$$

Assuming without loss of generality that $A_0 \leq 1$, we use the line above, $||\tilde{L}_i|| = 1$, the orthogonality of the $\{\tilde{L}_i, \tilde{f}_n\}$, and the disjoint support of the $\tilde{L}_i$ and g_n to see that

$$\begin{aligned}
A_0 ||\{d_k\}||_2^2 &= A_0 \left[\sum_{i \in B} d_i^2 + \sum_{n \in C} d_n^2 + \sum_{m \in D} d_m^2 \right] \\
&\leq \sum_{i \in B} d_i^2 + || \sum_{n \in C} d_n \tilde{f}_n + \sum_{m \in D} d_m g_m ||_2^2 \\
&= \int \sum_{i \in B} (d_i \tilde{L}_i)^2 + \int \sum_{n \in C} (d_n \tilde{f}_n)^2 + \int \sum_{m \in D} (d_m g_m)^2 + 2 \int \sum_{n \in C, m \in D} d_n d_m \tilde{f}_n g_m \\
&= \int (\sum_{i \in B, n \in C, m \in D} d_i \tilde{L}_i + d_n \tilde{f}_n + d_m g_m)^2 \\
&= || \sum_{i \in B} d_i \tilde{L}_i + \sum_{n \in C} d_n \tilde{f}_n + \sum_{m \in D} d_m g_m ||_2^2
\end{aligned}$$

A similar proof shows that

$$|| \sum_{i \in B} d_i \tilde{L}_i + \sum_{n \in C} d_n \tilde{f}_n + \sum_{m \in D} d_m g_m ||_2^2 \leq B_0 ||\{d_k\}||_2^2,$$

so $\{\tilde{L}_i, \tilde{f}_n, g_m\}$ is a Riesz basis of its span. Finally, by the Lemma 4.3 above, $\{L_i, f_n, g_m\}$ is a Riesz basis of V_0. $\Box$

We will now show that for some special cases, we need fewer edge functions than in other constructions. More precisely, when $A = 2$ and either $n(S) = 3$, $m = 2$, or $n(S) = 4$, $m = 3$, then we only need $n(S) - A = m - 1$ edge functions to preserve polynomial accuracy. The monomial t^{m-1} can be reproduced from the original $\phi^\ell(t - k)$ for $k \geq 0$ and $\phi_{L,n}$ for $n = 0, \ldots, m - 2$. The number of edge functions needed is smaller than the $n(S) = m$ needed in Daubechies [2] construction and the same as Meyer's [9] construction. In both of these cases, the number of scaling functions is $A = 1$. To write t^{m-1} in terms of $\phi^\ell(t - k)$, $k \geq 0$ and $\phi_{L,n}$, $n = 0, \ldots, m - 2$ we must find constants $\{\alpha_j\}$ for which

$$\sum_{j=0}^{m-2} \alpha_j \phi_{L,j}(t) + \sum_{j=1}^{2} \alpha_{j+m-2} \phi^j(t) = t^{m-1}$$

on $[0,1]$.

Rewriting this in terms of linearly independent $\bar{\phi}_k^\ell$ and using (8) we have

$$\sum_{j=0}^{m-2} \alpha_j \left[\sum_{\ell=1}^{2} \sum_{k=1-M_\ell}^{0} f_{j,k}^\ell \bar{\phi}_k^\ell \right] + \sum_{\ell=1}^{2} \alpha_{\ell+m-2} \bar{\phi}_0^\ell = \sum_{\ell=1}^{2} \sum_{k=1-M_\ell}^{0} f_{m-1,k}^\ell \bar{\phi}_k^\ell \tag{9}$$

on $[0,1]$.

To determine when these systems have solutions, we need the following well-known lemma (see for example [5]).

Lemma 4.5 *The constants* $\{f_{j,k}^\ell\}$ *satisfy the recurrance relation*

$$f_{j,k+1}^\ell = \sum_{i=0}^{j} \begin{pmatrix} j \\ i \end{pmatrix} f_{i,k}^\ell$$

for $\ell = 1, \ldots, A$ *and* $j = 0, \ldots, m-1$.

Remark. In particular, note that $f_{0,k}^\ell = f_{0,0}^\ell$ for $\ell = 1, \ldots, A$ and $k \in \mathbb{Z}$ and $f_{1,k}^\ell = f_{1,k+1}^\ell - f_{0,0}^\ell$ for $\ell = 1, \ldots, A$. Now for $A = 2$, we consider the following three cases:

Case I. $n(S) = 3$, $m = 2$. In this case, we can assume without loss of generality that $\operatorname{supp}(\phi^1) = [0,2]$ and $\operatorname{supp}(\phi^2) = [0,1]$. Using the linear independence of the translates of $\phi^\ell(\cdot - k)$, the system (9) becomes in matrix form:

$$\begin{bmatrix} f_{0,0}^1 & 1 & 0 \\ f_{0,0}^2 & 0 & 1 \\ f_{0,-1}^1 & 0 & 0 \end{bmatrix} \cdot \alpha = \begin{bmatrix} f_{1,0}^1 \\ f_{1,0}^2 \\ f_{1,-1}^1 \end{bmatrix} \tag{10}$$

Case II. $n(S) = 4$, $m = 3$, $\operatorname{supp}(\phi^1) = \operatorname{supp}(\phi^1) = [0,2]$. The system (9) becomes in matrix form:

$$\begin{bmatrix} \mathbf{f}_{0,0} & \mathbf{f}_{1,0} & I_2 \\ \mathbf{f}_{0,-1} & \mathbf{f}_{1,-1} & 0_2 \end{bmatrix} \cdot \alpha = \begin{bmatrix} \mathbf{f}_{2,0} \\ \mathbf{f}_{2,-1} \end{bmatrix} \tag{11}$$

where I_2 is the 2×2 identity matrix, 0_2 is the 2×2 zero matrix, and $\alpha = (\alpha_0, \ldots, \alpha_{m-1})^T$.

Case III. $n(S) = 4$, $m = 3$, $\operatorname{supp}(\phi^1) = [0,3]$ and $\operatorname{supp}(\phi^1) = [0,1]$. The system (9) becomes in matrix form:

$$\begin{bmatrix} \mathbf{f}_{0,0} & \mathbf{f}_{1,0} & 1 & 0 \\ f^1_{0,0} & f^1_{1,-2} & 0 & 0 \\ f^2_{0,0} & f^1_{1,-1} & 0 & 0 \end{bmatrix} \cdot \alpha = \begin{bmatrix} \mathbf{f}_{2,0} \\ f^1_{2,-2} \\ f^2_{2,-1} \end{bmatrix} \tag{12}$$

We have the following result for scaling vectors of length $A = 2$ whose components have short support.

Proposition 4.6 *There exists unique solutions to (10) and (12) and if the vectors* $\mathbf{f}_{0,0}$ *and* $\mathbf{f}_{1,0}$ *are linearly independent, then there exists a unique solution to (11). Thus in these cases, only* $m-1$ *edge functions are needed to produce polynomials of degree* m.

Proof.

Case I. Clearly the top two rows of the 3×3 matrix in (10) are linearly independent and from the support properties of ϕ^1, ϕ^2, we have

$$1 = \phi_{L,0}(0) = \sum_{\ell=1}^{2} \sum_{k=-1}^{0} f^\ell_{0,k} \bar{\phi}^\ell_k(0) = f^1_{0,-1} \bar{\phi}^1_{-1}(0)$$

so that $f^1_{0,-1} \neq 0$ and the 3×3 matrix in (10) must be nonsingular so that the system has a unique solution.

Case III. Clearly the top two rows of the 4×4 matrix in (12) are linearly independent. Thus in order for the matrix to be nonsingular, we must have $f^1_{1,-1} \neq f^1_{1,-2}$. Using the remark following Lemma 4.5, we see that

$$f^1_{1,-2} = f^1_{1,-1} + f^1_{0,0}$$

so that the nonsingularity of the matrix rests upon $f^1_{0,0} \neq 0$. But if $f^1_{0,0} = 0$, then

$$1 = f^2_{0,0} \sum_{k \in \mathbb{Z}} \phi^2(t-k).$$

But recall that $\phi^2(t)$ is supported only on $[0, 1]$ and is continuous so that it must be the case that $f^1_{0,0} \neq 0$.

Case II. Using the remark following Lemma 4.5, we see that $\mathbf{f}_{1,-1} = \mathbf{f}_{1,0} - \mathbf{f}_{0,0}$.

Certainly if $\mathbf{f}_{0,0}$ and $\mathbf{f}_{1,0}$ are linearly independent then the 4×4 matrix in (11) is nonsingular and the system thus has a unique solution. □

Remark. Whether the vectors $\mathbf{f}_{0,0}$ and $\mathbf{f}_{1,0}$ (given the regularity conditions imposed on Φ) are always linearly independent seems to be an open question.

We next return to the example of the scaling vector of Plonka and Strela [11] introduced in Section 2.

Example 4.7 *Note that from Example 4.2 that* $m = 3$ *and* $\mathbf{f}_{0,0} = (1,0)^T$, $\mathbf{f}_{1,0} = (\frac{1}{6}, 1)^T$ *are linearly independent so that by Proposition 4.6 Case II, we know that we need only construct two edge functions* $\phi_{L,0}$ *and* $\phi_{L,1}$ *in order to reproduce quadratics in* $V_0([0,\infty))$. *Using (8), we find that*

$$\phi_{L,0}(t) = \phi^2(t) + \phi^2(t+1)|_{t\geq 0}$$

and

$$\phi_{L,1}(t) = -\frac{1}{6}\phi^1(t) - \frac{1}{6}\phi^1(t+1) + \phi^2(t)|_{t\geq 0}.$$

The two edge functions are pictured below.

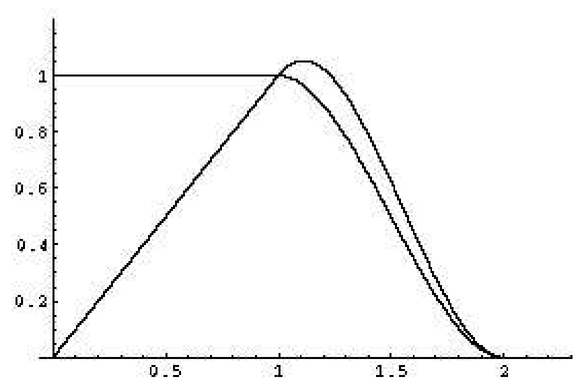

Fig. 5. The edge functions $\phi_{L,0}$ and $\phi_{L,1}$.

Our next example utilizes the scaling vector of Donovan, Geronimo, Hardin, and Massopust [3] from Example 3.2.

Example 4.8 *We return to the scaling vector of Example 3.2. Note that* $m =$

2 *and it is known that* $\mathbf{f}_{0,0} = (1, \sqrt{2})^T$. *Thus by (8), we see that*

$$\phi_{L,0}(t) = \phi^1(t) + \phi^1(t+1) + \sqrt{2}\phi^2(t)|_{t\geq 0}$$

The edge function is shown below.

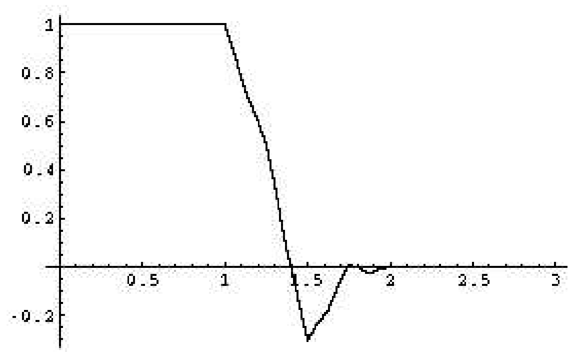

Fig. 6. The edge function $\phi_{L,0}$.

We now continue with a discussion on how to construct a scaling vector for $L^2([0,1])$ for the cases listed above. We assume that the edge functions $\phi_{L,0}, \ldots, \phi_{L,m-2}$, and $\phi_{R,0}, \ldots, \phi_{R,m-2}$ have been constructed. Then for M defined by (3), we select a resolution level j_0 so that $2^{j_0} \geq 2M$. This ensures that the support of the edge functions do not intersect. Start the basis for $V_{j_0}([0,1])$ by considering those functions $\phi^\ell(2^{j_0}t - k)$, $\ell = 1, 2$, $k \in \mathbb{Z}$ whose support is contained in $[0,1]$. If $s = M_1 + M_2$, then there are $2^{j_0+1} - s + 2$ interior functions. We then add the dilated versions of the edge functions $\phi_{L,0}(2^{j_0}t), \ldots, \phi_{L,m-2}(2^{j_0}t)$ and $\phi_{R,0}(2^{j_0}t), \ldots, \phi_{R,m-2}(2^{j_0}t)$ and take $V_{j_0}([0,1])$ to be the closed linear span of these functions. Note that the dimension of $V_{j_0}([0,1])$ is $2^{j_0+1} + 2m - s$. We can define spaces $V_n([0,1])$ at different resolutions in the standard manner. Note that the dimension of $V_n([0,1])$ is $2^{n+1} + 2m - s$. We conclude this section by constructing multiwavelets for the scaling vector of Plonka and Strela given in Example 3.3. Other multiwavelets for those scaling vectors falling into Cases I, II, or III can be constructed analogously.

Example 4.9 *Define the wavelet space*

$$W_{j_0}([0,1]) = V_{j_0+1}([0,1]) \ominus V_{j_0}([0,1])$$

Recall that $m = 3$ *and* $s = 4$ *so that the dimension of* $W_{j_0}([0,1])$ *is* $2^{j_0+2} + 2 - (2^{j_0+1} + 2) = 2^{j_0+1}$. *Since* $supp(\psi^1) = supp(\psi^2) = [0,3]$, *we see that we need interior basis elements*

$$\psi^\ell(2^{j_0}t - k)$$

where $\ell = 1,2$ *and* $k = 0, \ldots, 2^{j_0} - 3$. *We thus need*

$$2^{j_0+1} - 2(2^{j_0} - 2) = 4$$

edge functions. We describe how to construct the two left edge functions $\psi_{L,0}(t)$ *and* $\psi_{L,1}(t)$. *The construction of the right edge functions follows in a similar manner.*

For simplicity, let $g_i(t)$, $i = 1, \ldots, 8$ *denote the functions*

$$\phi_{L,0}(2t), \phi_{L,1}(2t), \phi^\ell(2t - k)$$

$\ell = 1,2$ *and* $k = 0,1,2$ *and* $f_j(t)$, $j = 1, \ldots, 6$ *denote the functions*

$$\phi_{L,0}(t), \phi_{L,1}(t), \phi^\ell(t - k)$$

$\ell = 1,2$ *and* $k = 0,1$. *We then seek constants* β_k^m, $k = 1, \ldots, 8$, $m = 0,1$ *such that*

$$\psi_{L,m}(t) = \sum_{k=1}^{8} \beta_k^m g_k(t)$$

with $< \psi_{L,m}, f_j >= 0$ *for* $m = 0,1$ *and* $j = 1, \ldots, 6$. *So we need two linearly independent solutions of*

$$A \cdot \beta^{\mathbf{m}} = 0$$

where $\beta = (\beta_1^m, \ldots, \beta_8^m)^T$ *and* A *is the* 6×8 *matrix with the* j,k *entry* $< g_k, f_j >$. *In the figures below, we graph* $\psi^1(t)$, $\psi^2(t)$ *as well as one possible choice for* $\psi_{L,0}(t)$ *and* $\psi_{L,1}(t)$.

5 Gibbs' Phenomenon for Positive Multiscaling Functions

We conclude this paper with a discussion of some results on Gibbs' phenomenon for scaling vectors in $L^2(\mathbb{R})$. In [14] the authors used a reproducing kernel to avoid Gibbs' phenomenon in wavelet expansions of functions in

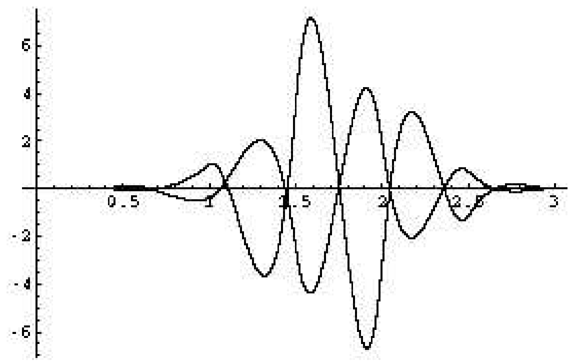

Fig. 7. The multiwavelets $\psi^1(t)$ and $\psi^2(t)$.

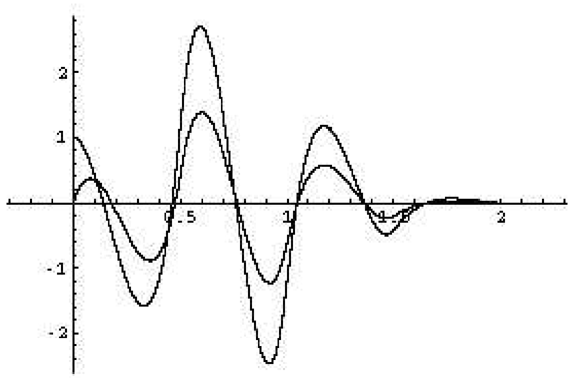

Fig. 8. The left edge functions $\psi_{L,0}$ and $\psi_{L,1}$.

$L^2(\mathbb{R})$. The kernel and the results were generalized to the case of scaling vectors and multiwavelets in [12]. In this section, we prove theorems demonstrating that Gibbs' phenomenon is a problem for many multiresolution analyses and we also sharpen our previous result for an important special case.

We classify multiresolution analyses into three categories:

(1) **Those with orthonormal bases.** In this case we can write

$$L^2(\mathbb{R}) = V_k \oplus (\oplus_{\ell \geq k} W_\ell)$$

where the direct sums are orthogonal, and the corresponding orthogonal projections P_k are defined by

$$P_k \left(\sum_{i=1}^{A} \sum_{j \in \mathbb{Z}} \alpha_{kj}^i \phi_{kj}^i + \sum_{\ell \geq k} \sum_{i=1}^{A} \sum_{j \in \mathbb{Z}} \beta_{\ell j}^i \psi_{\ell j}^i \right) = \sum_{ij} \alpha_{kj}^i \phi_{kj}^i \tag{13}$$

where $\phi_{kj}^i(t) = 2^{-k/2} \phi^i(2^k t - j) \quad$ for $i = 1, \ldots, A, \quad k, j \in \mathbb{Z}$.

(2) **Those with semi-orthogonal bases.** In this case the translates of the scaling function(s) are not orthogonal, but we can still write

$$L^2(\mathbb{R}) = V_k \oplus (\oplus_{\ell \geq k} W_\ell)$$

where the direct sums are orthogonal, and the corresponding orthogonal projections P_k are defined as in (13).

(3) **Those with non-orthogonal biorthogonal bases.** In this case the V_j and W_j spaces are non-orthogonal and

$$L^2(\mathbb{R}) = V_k \bigoplus \left(\bigoplus_{\ell \geq k} W_\ell \right)$$

where the direct sums $\bigoplus$ are not orthogonal, and the corresponding projections P_k defined as in (13) are not orthogonal. In this case, there is a dual multiresolution analysis with scaling vector Φ^* such that $< \phi_{kj}^i, {\phi^*}_{mn}^{\ell} >= \delta_{i\ell}\delta_{km}\delta_{jn}$, $k, j, m, n \in \mathbb{Z}$, $i, \ell = 1, \ldots, A$.

Here is a precise definition of Gibbs' phenomenon.

Definition 5.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a square integrable bounded function with a jump discontinuity at 0: the limits $\lim_{x \to 0^+} f(x) = f(0+)$ and $\lim_{x \to 0^-} f(x) = f(0-)$ exist and are different. Without loss of generality we assume $f(0+) > f(0-)$. Suppose we have a multiresolution analysis of $L^2(\mathbb{R})$ with multiresolution spaces (V_j) generated by a scaling vector. We say a sequence of operators (L_j), $L_j : L^2(\mathbb{R}) \to V_j$ is* **admissible** *if $\lim_{j \to \infty} L_j(f) = f$ in the L^2 sense, for all $f \in L^2(\mathbb{R})$. We say that a wavelet expansion of f with respect to a scaling vector and an admissible sequence (L_j) shows a Gibbs' phenomenon at 0 if there is a positive sequence (x_m) with $\lim_{m \to \infty} x_m = 0$ and $\lim_{m \to \infty} L_m(x_m) > f(0+)$, or if there is a negative sequence (t_m) with $\lim_{m \to \infty} t_m = 0$ and $\lim_{m \to \infty} L_m(t_m) < f(0-)$.*

Observe that we do not require the maps L_j to be orthogonal projections since many interesting MRA's are built from Riesz or biorthogonal bases, rather than orthogonal bases. Moreover, we shall see that we can avoid Gibbs' by taking an admissible sequence of operators that are not even projections.

The definition is otherwise quite standard.

Our main result is to show that nearly all interesting scaling vectors generating multiresolution analyses will suffer from Gibbs' phenomenon. More precisely, we prove the theorem below.

Theorem 5.2 *Let $\Phi = (\phi^1, \ldots, \phi^A)^T$ be a scaling vector satisfying (A1) with approximation order at least 2. If the multiresolution analysis is orthogonal or Φ has a dual biorthogonal basis Φ^* that is compactly supported, then the corresponding wavelet expansion shows a Gibbs' phenomenon at least one side of 0.*

To prove this result, we modify and generalize Shim and Volkmer's [13] approach for the single scaling function orthonormal case in two directions: to include biorthogonal bases and to include multiple scaling functions. We are also able to replace a pair of rather technical derivative and decay hypotheses in [13] with the hypotheses on compact support and approximation order. We now state their main result from [13].

Theorem 5.3 (Shim,Volkmer) *Let ϕ be a continuous scaling function generating an orthonormal multiresolution analysis that is differentiable at a dyadic number with nonvanishing derivative there, and that satisfies*

$$|\phi(t)| \leq K(1+|t|)^{-\beta} \text{for } t \in \mathbb{R}$$

with constants $K > 0$ and $\beta > 3$. Then the corresponding wavelet expansion shows a Gibbs phenomenon at one side of 0.

Before we present the proof to Theorem 5.2, we first introduce some notation and state and prove two lemmas.

Let Q_m denote the projection map onto the space V_m defined above in (13).

Define the reproducing kernel $q(s,t)$ by

$$q(s,t) = \sum_{i=1}^{A} \sum_{j \in \mathbb{Z}} \phi^i(s-j)\phi^{*i}(t-j) \tag{14}$$

and q_m by $q_m(s,t) = 2^m q(2^m s, 2^m t)$, where (ϕ^{*i}) is the biorthogonal basis. Observe that

$$(Q_0 f)(s) = \sum_{i=1}^{A} \sum_{j \in \mathbb{Z}} \left\langle f, \phi^{*i}(\cdot - j) \right\rangle \phi^i(s-j) = \int_{\mathbb{R}} f(t) q(s,t) dt$$

$\forall f \in L^2(\mathbb{R})$.

Finally, let H denote the Heaviside function and define function r by

$$r = H - Q_0 H$$

Lemma 5.4 *The coefficients c_i in (A2) satisfy*

$$c_i = \int_{\mathbb{R}} \phi^{*i}(t)dt$$

and for $m \in \mathbb{Z}$,

$$\int_{\mathbb{R}} q_m(s,t)dt = 1.$$

Proof. First observe that from the biorthogonality and (A2), we have

$$\int_{\mathbb{R}} \phi^{*i}(t)dx = \int_{\mathbb{R}} \phi^{*i}(t) \sum_{\ell=1}^{A} \sum_{k\in\mathbb{Z}} c_\ell \phi^\ell(t-k)dt = c_i$$

The second result follows from integrating (14) with respect to t and applying our formula for c_i and (A2). □

Lemma 5.5 *Let $\Phi = (\phi^1, \ldots, \phi^A)^T$ be a compactly supported, continuous scaling vector with approximation order at least 2 generating a multiresolution analysis for $L^2(\mathbb{R})$. If the multiresolution analysis is orthogonal or Φ has a dual biorthogonal basis Φ^* that is compactly supported then the following are true:*

(1) $Q_0 H = H - r$ is continuous,
(2) $r(t)$ is compactly supported and continuous, except for a jump discontinuity at 0.
(3) $r \in \bigoplus_{j\geq 0} W_j^*$
(4) $\int_{\mathbb{R}} tr(t)dt = 0$.

Proof.

1. First note that $(Q_0H)(s) = \int_{\mathbb{R}} H(t)q(s,t)dt = \sum_{n,\ell} \phi^\ell(s-n)d_{n,\ell}$ where

$$d_{n,\ell} = \int_0^\infty \phi^{*\ell}(t-n)dt - \int_{-\infty}^0 \phi^{*\ell}(t-n)dt.$$

Each $\phi^\ell(\cdot - n)$ is continuous and compactly supported, so Q_0H is continuous.

2. r is continuous except for a jump discontinuity at 0. This follows from Part 1 and the fact that $r = H - Q_0H$. Thus it suffices to show that r has compact support. To this end, observe that for $t \geq 0$, Lemma 5.4 tells us that

$$\begin{aligned} r(t) &= 1 - \int_{\mathbb{R}} q(t,y)H(y)dy \\ &= 2\int_{-\infty}^{0} q(t,y)dy \end{aligned}$$

Similarly for $t < 0$, $r(t) = -2\int_0^\infty q(t,y)dy$.

Now by the compact support of the ϕ^ℓ and $\phi^{*\ell}$, for $t > M$, where M is given by (3) we have

$$r(t) = 2\sum_{\ell=1}^{A}\sum_{n\geq 0}\phi^\ell(t-n)\int_{-\infty}^{0}\phi^{*\ell}(y-n)dy = 0.$$

Let M^* be defined by (3) for the dual scaling vector Φ^*. Then for $t < -M^* - M$

$$r(t) = 2\sum_{\ell=1}^{A}\sum_{n\leq -M}\phi^\ell(t-n)\int_{0}^{\infty}\phi^{*\ell}(y-n)dy = 0,$$

whence $r(t)$ has compact support.

3. Next, for arbitrary $j = 1, \ldots, A$ and $k \in \mathbb{Z}$, observe that

$$\begin{aligned} \int_{\mathbb{R}} r(t)\phi^{*j}(t-k)dt &= \int_{\mathbb{R}} H(t)\phi^{*j}(t-k)dx - \int_{\mathbb{R}}(Q_0H)(t)\phi^{*j}(t-k)dt \\ &= \int_{\mathbb{R}} H(t)\phi^{*j}(t-k)dt - \int_{\mathbb{R}}\int_{\mathbb{R}}(H(y)q(t,y)dy)\phi^{*j}(t-k)dt \end{aligned}$$

which can be expressed as

$$\begin{aligned} &= \int_{\mathbb{R}} H(t)\phi^{*j}(t-k)dt - \int_{\mathbb{R}}\int_{\mathbb{R}} H(y)\sum_{m=1}^{A}\sum_{n\in\mathbb{Z}}\left[\phi^m(t-n)\phi^*_m(y-n)\right]\phi^{*j}(t-k)dydt \\ &= \int_{\mathbb{R}} H(t)\phi^{*j}(t-k)dt - \int_{\mathbb{R}} H(y)\sum_{m=1}^{A}\sum_{n\in\mathbb{Z}}\left(\int_{\mathbb{R}}\phi^m(t-n)\phi^{*j}(t-k)dt\right)\phi^*_m(y-n)dy \end{aligned}$$

$$= \int_{\mathbb{R}} H(t)\phi^{*j}(t-k)dt - \int_{\mathbb{R}} H(y)\cdot\phi^{*j}(y-k)dy$$

so $r \perp \Phi^*_{jk}$ $j = 1, \ldots, A$ and $k \in \mathbb{Z}$. Writing $L^2(\mathbb{R}) = V_0^* \oplus \left(\bigoplus_{k\geq 0} W_k^*\right)$ we must have $r \in \bigoplus_{k\geq 0} W_k^*$.

4. Part 3 tells us that $r = \sum_{\ell\geq 0} \alpha_{i,\ell j}\psi^*_{i,\ell j}$ where the $\psi^*_{i,\ell j} \in W^*_\ell$ are the multiwavelets of the dual basis. Since Φ has approximation order at least 2, $t = \sum_{n,\ell} \beta_{n\ell}\phi^\ell(t-n)$ for some $(\beta_{n\ell})$ so

$$\int_{\mathbb{R}} tr(t)dt = \int_{\mathbb{R}} \left\{\sum_{n,\ell} \beta_{n\ell}\phi^\ell(\cdot - n)\right\}\left\{\sum \alpha^l_{ij}\psi^\ell_{ij}\right\} = 0$$

since $V_0 \perp W_j^*$ for each $j \geq 0$, $j \in \mathbb{Z}$. □

Now we are ready to prove Theorem 5.2.

Proof of Theorem 5.2. We first claim that $r(t_1) < 0$ for some $t_1 > 0$ or $r(t_2) > 0$ for some $t_2 < 0$. For otherwise $\int_{\mathbb{R}} tr(t)dt = 0$ would force $r(t) = 0$ almost everywhere. This is impossible by Part 2 of Lemma 5.5.

Now consider the case $r(t_1) < 0$ for some $t_1 > 0$. Then $r(t_1) = 1 - \int_{\mathbb{R}} q(t_1, y)H(y)dy < 0$ implies that

$$\int_{\mathbb{R}} q(t_1, y)H(y)dy > 1. \tag{15}$$

We now show there must be a Gibbs' phenomenon for the Haar wavelet

$$h(t) = \begin{cases} 1 \text{ if } & 0 \leq t \leq 1 \\ -1 \text{ if } & -1 \leq t < 0 \end{cases}$$

Clearly $\lim_{m\to\infty} t_1 2^{-m} = 0$, but

$$\begin{aligned}\lim_{m\to\infty}(Q_m h)(t_1 2^{-m}) &= \lim_{m\to\infty} \int_{\mathbb{R}} 2^m q(t_1, 2^m y)h(y)dy \\ &= \lim_{m\to\infty} \int_0^1 2^m q(t_1, 2^m y)dy - \int_{-1}^0 2^m q(t_1, 2^m y)dy \\ &= \lim_{m\to\infty} \int_0^{2^m} q(t_1, t)dt - \int_{-2^m}^0 q(t_1, t)dt\end{aligned}$$

$$= \int_{-\infty}^{\infty} q(t_1, t)H(t)dt > 1$$

by (15). Thus h exhibits Gibbs phenomenon at 0. The case $r(t_2) > 0$ for some $t_2 < 0$ is similar. □

We conclude this section by showing that we can avoid Gibbs by using a special reproducing kernel. Of course, the reproducing kernel here corresponds to map into V_m that is not a projection. Note that in the theorem below the compact support and positivity together allow a improved statement over our previous result (Proposition 3.7 of [12]) and that of Shen and Walter (Proposition 4.3 of [14]): we can specify the resolution of the kernel and can give a tighter upper bound on the approximation in V_m.

Before we present a sharper version of Proposition 3.6 that appeared in [12], let us add the following assumption:

(A5) Assume that for each ϕ^j with $c_j \neq 0$ we have $\phi^j \geq 0$.

Note that under (A5),

$$K(s,t) = \sum_{j=1}^{A} \sum_{k \in \mathbb{Z}} \left(\frac{c_j}{\int_{\mathbb{R}} \phi^j} \right) \phi^j(t-k)\phi^j(s-k).$$

For the sake of notation, we define $K_m(s,t)$ by

$$K_m(s,t) = 2^m K(2^m s, 2^m t).$$

Proposition 5.6 *Under assumptions (A1) - (A5) on the scaling vector,*

(1) $\int_{\mathbb{R}} K_m(s,t)ds = 1 \qquad \forall m \in \mathbb{Z}, t \in \mathbb{R}$
(2) $K_m(s,t) \geq 0 \qquad \forall m \in \mathbb{Z}, t \in \mathbb{R}$
(3) For each $\gamma > 0$, *if* $m > \log_2(\frac{M}{\gamma})$ *then* $\sup_{|s-t|>\gamma} K_m(s,t) = 0$.

Proof. The proofs of 1. and 2. are identical to those of Proposition 3.6 of [12] and are thus omitted. To see 3., observe that $|\mathrm{supp}\,(\phi^j(2^m \cdot -k))\,| \leq M2^{-m} < \gamma$. So if $|t-s| > \gamma$ then $\phi^j(2^m s-k)\phi^j(2^m t-k) = 0\ \forall k \in \mathbb{Z}$. Thus

$$\sup_{|s-t|>\gamma} K_m(s,t) = 2^m \sum_{j=1}^{A} \sum_{k \in \mathbb{Z}} \left(\frac{c_j}{\int_{\mathbb{R}} \phi^j} \right) \sup_{|s-t|>\gamma} \phi^j(2^m s-k)\phi^j(2^m t-k) = 0. \qquad \square$$

Theorem 5.7 *Let* $\Phi = (\phi^1, \ldots, \phi^A)^T$ *be a scaling vector and assume (A1) - (A5) hold. Suppose that* $M_1 \leq f(t) \leq M_2$ *on* $[a,b]$. *Then for each* $\delta > 0$ *and* $m > \log_2\left(\frac{M}{\delta}\right)$,

$$M_1 \leq f_m(t) \leq M_2$$

whenever $t \in (a+\delta, b-\delta)$. *Here,* $f_m \in V_m$ *where*

$$f_m(t) = \int_{\mathbb{R}} K_m(s,t) f(s) ds.$$

Proof. For $t \in (a+\delta, b-\delta)$ choose $m > \log_2\left(\frac{M}{\delta}\right)$ and write $f_m(t)$ as

$$\begin{aligned} f_m(t) &= \int_{\mathbb{R}} K_m(s,t) f(s) ds = \left(\int_{-\infty}^{a} + \int_{a}^{b} + \int_{b}^{\infty} \right) K_m(s,t) f(s) ds \\ &\leq 2 \sup_{|s-t|>\delta} K_m(s,t) \int_{\mathbb{R}} |f(s)| ds + M_2 \int_{\mathbb{R}} K_m(s,t) ds \\ &= M_2 \end{aligned}$$

using the Proposition 5.6 above. The proof that $M_1 \leq f_m(t)$ is similar. □

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