On the Support Properties of Scaling Vectors

Peter R. Massopust David K. Ruch*
Patrick J. Van Fleet†
Sam Houston State University, Huntsville, TX 77341

Abstract

In Chui and Wang [3], support properties are derived for a scaling function generating a function space $V_0 \subseteq L^2(\mathbb{R})$. Motivated by this work, we consider support properties for scaling vectors. In [9], Goodman and Lee derive necessary and sufficient conditions for the scaling vector $\{\phi_1, \ldots, \phi_r\}$, $r \geq 1$, to form a Riesz basis for V_0 and develop a general theory for spline wavelets of multiplicity $r > 1$. We consider conditions under which linear combinations of scaling functions generate V_0. These conditions also characterize all other scaling vectors that generate the same V_0. In addition we describe the scaling vectors of minimal support for V_0.

Next, we give sufficient conditions on the two-scale symbol for scaling vectors under which a given matrix refinement equation can be solved. A spline-wavelet example illustrates these results.

For the single scaling function ϕ, the support of ϕ is characterized by the degree of the two-scale symbol. The situation is more complicated in the scaling vector case. We prove a result that gives the support of the scaling vector under certain conditions on the coefficient matrices. This result is illustrated by an example of fractal wavelets derived in Geronimo, Hardin, and Massopust [8].

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1 Introduction

In wavelet theory, a scaling function ϕ is a function that along with its integer translates $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis for $V_0 \subset L^2(\mathbb{R})$. Recall that the existence of such a basis

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for V_0 is one of five requirements that must be satisfied in order for the ladder of closed subspaces $\cdots \subset V_1 \subset V_0 \subset V_{-1} \cdots$ to form a multiresolution analysis (see Daubechies [6]). If such a multiresolution analysis exists then Daubechies [6] proved the existence of a wavelet ψ that along with its translates and dilates form an orthonormal basis for $L^2(\mathbb{R})$.

Wavelets can be constructed to possess many desirable properties for applications. Perhaps the three most important properties are orthogonality, regularity, and compact support. Chui [2] and Chui and Wang [3] investigated support and regularity properties by associating with a scaling function ϕ its two-scale symbol $P_\phi(\omega)$ and then imposing certain admissibility conditions onto $P_\phi(\omega)$. In a certain sense they have shown that the symbol $P_\phi(\omega)$ carries all information necessary to characterize its scaling function ϕ.

It is our intent in this paper to investigate the properties given above for a *scaling vector*. Scaling vectors were first studied in [8] where the authors assumed that the integer translates of $\phi_1, \ldots, \phi_r$, $r \geq 1$ formed a Riesz basis for V_0. For completeness we follow Geronimo, Hardin, and Massopust [8] and define a *multiresolution analysis* (MRA) for closed subspaces of $\{V_k\}_{k \in \mathbb{Z}} \subseteq L^2(\mathbb{R})$ below:

Let N be an integer greater than 1 and let $\{\phi_j : \ j = 1, \ldots, r\}$ be a given collection of functions in V_0. The ladder of spaces $\{V_k\}_{k \in \mathbb{Z}}$ is said to form a *multiresolution analysis* of $L^2(\mathbb{R})$ if and only if

- Nestedness. $V_{k+1} \subseteq V_k$, $k \in \mathbb{Z}$
- Separation. $\cap_{k \in \mathbb{Z}} V_k = \{0\}$
- Density. $\overline{\cup V_k} = L^2(\mathbb{R})$
- $f \in V_k \iff f(N\cdot) \in V_{k-1}$
- The set $B_{\boldsymbol{\phi}} = \{\phi_j(\cdot - l) : j = 1, \ldots r, l \in \mathbb{Z}\}$ is a Riesz basis for V_0.

In [11], the authors proved that the separation and the density properties hold if the refinable functions ϕ_j, $j = 1, \ldots, r$ are in $L^2(\mathbb{R})$. Necessary and sufficient conditions for $B_{\boldsymbol{\phi}}$ to form a Riesz basis can be found in [8, 9, 14]. In [9] the authors placed special emphasis on the case where each ϕ_i belongs to a certain spline space.

We recall that Theorem 3.2 in [8] states that $B_{\boldsymbol{\phi}}$ is a Riesz basis for V_0 if and only if the $r \times r$ matrix

$$E_{\boldsymbol{\phi}}(\omega) = \sum_{k=-\infty}^{\infty} \hat{\boldsymbol{\phi}}(\omega + 2\pi k) \hat{\boldsymbol{\phi}}^*(\omega + 2\pi k) \tag{1}$$

is nonsingular. Here

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_r \end{bmatrix}$$

and

$$\hat{\boldsymbol{\phi}}(\omega) = \int_{\mathbb{R}} e^{-i\omega x} \boldsymbol{\phi}(x) dx$$

denotes the Fourier transform of the vector $\boldsymbol{\phi}$.

At this time let us introduce some more notation and conventions.

We will make use of the following N-scale symbol in the sequel.

$$P_{\boldsymbol{\phi}}(\omega) = P(\omega) = \frac{1}{N} \sum_k \mathbf{C}_k z^k$$

where $z = e^{-i\omega/N}$ and the $r \times r$ matrices $\mathbf{C}_k$ satisfy the *matrix refinement equation*:

$$\boldsymbol{\phi}(x) = \sum_k \mathbf{C}_k \boldsymbol{\phi}(Nx - k). \tag{2}$$

The Fourier transform formulation of (2) corresponds to the equation:

$$\hat{\boldsymbol{\phi}}(\omega) = P(\frac{\omega}{N}) \hat{\boldsymbol{\phi}}(\frac{\omega}{N}). \tag{3}$$

As evidenced by the work in [8, 9, 14] and possibly elsewhere, scaling vectors allow for more flexibility when attempting to construct functions that are orthogonal, compactly supported and of some desired regularity. Another advantage of scaling vectors is that they allow for a broader choice of V_0. For example, using two scaling functions, Goodman and Lee [9] have constructed wavelets whose multiresolution analysis is built from spaces of C^1 cubic splines. These spaces are quite popular in many applications. One disadvantage of the scaling vector approach is the larger number of computations that will be performed in applications. Perhaps the greatest difficulty in working with scaling vectors is the fact that the well known refinement equation obeyed by a single scaling function becomes a matrix refinement equation in the scaling vector setting. Note that the coefficients in (2) are matrices so commutativity in general is not guaranteed. The analysis in the vector case is harder since one deals with infinite products of matrices rather than scalars.

We will show that it is the action of this matrix symbol $P_{\boldsymbol{\phi}}(\omega)$ on an eigenvector associated with the eigenvalue 1 of $P_{\boldsymbol{\phi}}(0)$ that determines compactness and length of support properties for $\boldsymbol{\phi}$. We will also give a theorem that shows the existence of a solution to the matrix refinement equation and by means of an example show that these results hold even if the infinite matrix product $\prod_k P_{\boldsymbol{\phi}}(\frac{\omega}{2^k})$ does not converge. (After completion of this paper, the authors learned of work by Heil and Colella [5] dealing with the existence and uniqueness of distributional solutions to matrix refinement equations. However their approach is different and their results are distinct.)

The notion of analyzing the effect of applying $P_{\boldsymbol{\phi}}(\omega)$ to an eigenvector u^* is a natural one and we see from the following proposition how it relates to the single scaling function case.

Convention. Throughout the sequel we assume that $B_{\boldsymbol{\phi}} \subseteq L^2(\mathbb{R})$ forms a basis for V_0.

Proposition 1.1 *Assume that ϕ satisfies (2) and suppose $\hat{\phi}(0) \neq 0$. Then the matrix $P(0)$ has spectral radius $\sigma_{P(0)} = 1$. Furthermore, there exists a nonzero vector u such that*

$$u^* \left(\frac{1}{N} \sum \mathbf{C}_k \right) = u^* \tag{4}$$

and

$$u^* P(2\pi j) = 0, \; j = 1, \ldots, N-1. \tag{5}$$

Proof. Observe from (3) that $P(0)$ has eigenvalue 1. Thus the spectral radius $\sigma_{P(0)} \geq 1$. Let y^* be a left eigenvector of $P(0)$ corresponding to an eigenvalue ρ such that $|\rho| = \sigma_{P(0)}$. As a trivial corollary of [8] Theorem 3.3(a), we have

$$E_{\phi}(\omega) = \sum_{j=0}^{N-1} P((2\pi j + \omega)) E_{\phi}((2\pi j + \omega)/N) P^*(2\pi j + \omega) \tag{6}$$

Setting $\omega = 0$ and multiplying (6) on the left by y^* and on the right by y we obtain:

$$y^* E_{\phi}(0) y = y^* P(0) E_{\phi}(0) P^*(0) y + \sum_{j \neq 0} y^* P(2\pi j) E_{\phi}(2\pi j/N) P^*(2\pi j) y.$$

Since $y^* P(0) = \rho y^*$, we have

$$(1 - |\rho|^2) y^* E_{\phi}(0) y = \sum_{j \neq 0} y^* P(2\pi j) E_{\phi}(2\pi j/N) P^*(2\pi j) y. \tag{7}$$

Noting from (1) that E_{ϕ} is positive definite, yields

$$(1 - |\rho|^2) y^* E_{\phi}(0) y \geq 0.$$

Thus $|\rho| = 1$.

To prove the remainder of the proposition, let u^* be a left eigenvector of $P(0)$ associated with the eigenvalue 1. We observe that for $\omega = 0$

$$u^* \left(\frac{1}{N} \sum \mathbf{C}_k \right) = u^*.$$

To prove (5), replace y with u in (7). Then

$$0 = \sum_{j \neq 0} u^* P(2\pi j) E_{\phi}(2\pi j/N) P^*(2\pi j) u.$$

Since each term must be nonnegative and E_{ϕ} is nonsingular, it must be that $u^* P(2\pi j) = 0$. $\square$

We note that (4) is analogous to the scalar case (see for example [2, 6]).

$$\frac{1}{N} \sum_k p_k = 1.$$

Furthermore, when $N = 2$, (5) is analogous to the scalar condition (see [2, 6])

$$\frac{1}{2}\sum_k (-1)^k p_k = 0.$$

Other similarities exist between the single scaling function and the scaling vector. In the single scaling function case it is known that if ϕ_1 and ϕ_2 are scaling functions for V_0 and ϕ_1 is minimally supported then ϕ_2 is a finite linear combination of ϕ_1 and its integer translates. We will give an analogous result for scaling vectors. In addition we obtain a result related to the minimality of the support of a scaling vector.

In the scalar case, the number of nonzero terms in the two-scale symbol $P_{\boldsymbol{\phi}}(\omega)$ gives the length of the support of the associated scaling function ϕ. A similar result holds for a scaling vector, provided that the first and last matrices in the corresponding symbol are not nilpotent. Examples will illustrate this theorem. One of the examples involves the class of scaling vectors constructed in [8], i.e., scaling vectors whose components are piecewise fractal interpolation functions.

The remainder of this paper is organized as follows. In Section 2 we state and prove a result that relates $P_{\boldsymbol{\phi}}(\omega)$ to the existence of a scaling vector for V_0. An example showing that $\prod_k P_{\boldsymbol{\phi}}(\frac{\omega}{2^k})$ need not converge is also included. We conclude the next section with a result that shows how to iteratively construct $\boldsymbol{\phi}$. In Section 3 we characterize scaling vectors for V_0. We also report results that relate to the minimality and length of the support of scaling vectors. Examples are included to illustrate our results.

2 Sufficient Conditions for Solving the Refinement Equation.

The results in this section are largely motivated by the work of C.K. Chui [2], A. Cohen [4], and I. Daubechies [6]. In [2], Chui defines an *admissible two-scale symbol* and consequently obtains sufficient conditions for the solution of the two-scale relation. He also shows that under some conditions, solutions of the two-scale relation are in $L^2(\mathbb{R})$ and that the degree of the two-scale symbol gives the length of the scaling function's support.

We examine these ideas in the multiple scaling function setting, beginning the section with a result giving sufficient conditions for the existence of a solution to the matrix refinement equation. Additional conditions can be imposed to guarantee that such a solution is in $L^2(\mathbb{R})$. Finally, we address the relationship between the degree of the two scale symbol and the scaling vector's support.

Recall that the two-scale symbol $P(\omega)$ is given by

$$P(\omega) = \frac{1}{N}\sum_{k\in\mathbb{Z}} \mathbf{C}_k z^k,$$

where $z = \exp(-i\omega/N)$, and assume that u is a right eigenvector of the matrix $P(0)$ associated with eigenvalue 1. For ease of notation, we set $P_k = P(\frac{\omega}{2^k})$, $k \in \mathbb{Z}$. Throughout the sequel, we take $N = 2$.

Theorem 2.1 *If there exists $C_1 > 0$ and $0 < \alpha \leq 1$ such that*

$$||u - P_k u||_{L^2} \leq C_1 \left(\frac{|\omega|}{2^k}\right)^\alpha \quad \textit{for large } k, \tag{8}$$

and if $P(0)$ has spectral radius 1, then

$$\lim_{n\to\infty} \left((\prod_{k=1}^{n} P_k) u \right) := g(\omega) = \begin{bmatrix} g_1(\omega) \\ \vdots \\ g_r(\omega) \end{bmatrix} \tag{9}$$

converges pointwise and $g(\omega)$ satisfies

1) $g(\omega) = P(\frac{\omega}{2}) g(\frac{\omega}{2})$

2) $g(0) = u$.

Furthermore, if $g_i \in L^2(\mathbb{R})$, $i = 1, \ldots, r$, then there exists $\boldsymbol{\phi}$ satisfying (2) with $\phi_i \in L^2(\mathbb{R})$, $i = 1, \ldots, r$ and $\hat{\boldsymbol{\phi}} = g$.

Remark 1. We note that it is sufficient to verify that g_i satisfies the following growth condition

$$|g_i(\omega)| \leq C_2 (1 + |\omega|)^\eta, \quad \text{for all} \ \ \omega, \tag{10}$$

for some $C_2 > 0$ and $\eta < -\frac{1}{2}$ to ensure that $g_i \in L^2(\mathbb{R})$.

Remark 2. Throughout the sequel, we will drop the parentheses on the product in (9). It will be understood that we will first compute the n-fold product, next multiply by the vector u, and finally take the limit as $n \to \infty$. Note that we are not requiring the infinite product of the P_k matrices to converge. An example will be provided after the proof of Theorem 2.1 to illustrate that convergence of the infinite product of matrices is not needed.

Proof. We begin with the following identity that can be established by induction. For $M \geq L \geq 1$,

$$u - \prod_{k=L}^{M} P_k u = \sum_{j=L}^{M} (\prod_{k=L}^{j-1} P_k)(u - P_j u) \tag{11}$$

We adopt the convention that

$$\prod_{k=L}^{L-1} P_k = I.$$

Since the spectral radius of $P(0)$ is 1 and since P_k is pointwise convergent to $P(0)$, we can find for each ω a $\gamma < 2^\alpha$ and a sufficiently large L so that

$$||P_k||_{L^2} \leq \gamma < 2^\alpha$$

for all $k \geq L$. We then use (8) and (11) to obtain

$$\begin{aligned} ||u - \prod_{k=L}^{M} P_k u||_{L^2} &\leq \sum_{j=L}^{M} \prod_{k=L}^{j-1} ||P_k||_{L^2}\, C_1 \left(\frac{|\omega|}{2^j}\right)^\alpha \\ &\leq \frac{C_1|\omega|^\alpha}{2^{L\alpha}} \sum_{k=0}^{M-L} \left(\frac{\gamma}{2^\alpha}\right)^k. \end{aligned} \tag{12}$$

Letting $M \to \infty$, we see from (12) that

$$\lim_{M\to\infty} \prod_{k=L}^{M} P_k u$$

converges for all ω. Thus

$$\lim_{M\to\infty} \prod_{k=1}^{M} P_k u := g(\omega)$$

converges for all ω.

To prove 1), note that

$$\begin{aligned} P(\frac{\omega}{2})g(\frac{\omega}{2}) &= P(\frac{\omega}{2}) \lim_{M\to\infty} \prod_{k=2}^{M} P_k u - P(\frac{\omega}{2}) \prod_{k=2}^{n} P_k u + P(\frac{\omega}{2}) \prod_{k=2}^{n} P_k u \\ &= P(\frac{\omega}{2}) \left(\lim_{M\to\infty} \prod_{k=2}^{M} P_k u - \prod_{k=2}^{n} P_k u \right) + \prod_{k=1}^{n} P_k u. \end{aligned}$$

Now let $n \to \infty$ on the right hand side to obtain

$$P(\frac{\omega}{2}) \cdot 0 + \lim_{n\to\infty} \prod_{k=1}^{n} P_k u = g(\omega).$$

Recalling that $u = P(0)u$ and evaluating (9) at $\omega = 0$ yields

$$g(0) = u.$$

To complete the proof, note that if each component g_i of g is in $L^2(\mathbb{R})$ then by the isometry of the Fourier transform, we know there exists $\boldsymbol{\phi}_i \in L^2(\mathbb{R})$ such $\hat{\boldsymbol{\phi}} = g$. $\square$

We now consider an example that illustrates that the matrix

$$P_\infty(\omega) = \prod_{k=1}^{\infty} P_k \tag{13}$$

need not converge in order to establish the existence of $g(\omega)$.

Example 2.2 *Let*

$$P(\omega) = \left(\frac{1+z}{2}\right)^M \begin{bmatrix} 1 & 0 \\ \frac{1-z}{2} & -1 \end{bmatrix}, \tag{14}$$

with $z = e^{-i\omega/2}$ *and* $M \geq 1$*. Then* P_∞ *does not exist,* $g(\omega) = \lim_{n\to\infty} \prod_{k=1}^n P_k u \in L^2(\mathbb{R})$ *and* $\phi = \hat{g} \in C^{(\alpha)}(\mathbb{R})$*, for* $\alpha < M - 1$*.*

Proof. Note that

$$\prod_{k=1}^n P_k = \prod_{k=1}^n \left(\frac{1+z_k}{2}\right)^M \begin{bmatrix} 1 & 0 \\ k_n & (-1)^n \end{bmatrix},$$

where $k_n = k_{n-1} + (-1)^{n-1}(\frac{1-z_n}{2})$. Here, $z_k = \exp(-i\omega/2^k)$. Observe that this recursive expression gives

$$k_n = \sum_{j=1}^n (-1)^{j-1} \left(\frac{1-z_j}{2}\right) = \sum_{j=1}^n (-1)^{j-1} \left(\frac{1-\cos(\omega/2^j)}{2} + \frac{i\sin(\omega/2^j)}{2}\right).$$

For each ω, $\lim_{n\to\infty} k_n$ exists since the real and imaginary parts both converge.
For $|\omega| \geq 1$, there exists J such that $2^J \leq |\omega| \leq 2^{J+1}$. For $n > J+1$ we have

$$\begin{aligned}
|k_n| &\leq \sum_{j=1}^n \left|\frac{1-z_j}{2}\right| \\
&= \sum_{j=1}^n |\sin(\omega/2^{j+1})| \\
&\leq \sum_{j=1}^{J-1} |\sin(\omega/2^{j+1})| + \sum_{j=J}^n |\omega|/2^{j+1} \\
&\leq (J-1) + |\omega|/2^J \\
&\leq J+1 \\
&\leq \log_2(|\omega|) + 1.
\end{aligned}$$

For $|\omega| < 1$,

$$\begin{aligned}
|k_n| &\leq \sum_{j=1}^n |\sin(\omega/2^{j+1})| \\
&\leq \sum_{j=1}^\infty |\omega|/2^{j+1} \\
&< 1.
\end{aligned}$$

Since $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$g_2(\omega) = \lim_{n\to\infty} \prod_{k=1}^n \left(\frac{1+z_k}{2}\right)^M \cdot k_n,$$

and we have

$$|g_2(\omega)| \leq \operatorname{sinc}^M(\omega/2)\,(\log_2(|\omega|+1)+1) \leq C\,\frac{\log(1+|\omega|)+1}{(1+|\omega|)^M}.$$

Hence $g_2 \in L^2(\mathbb{R})$, and there exists $\phi_2 \in L^2(\mathbb{R})$ such that $\hat{\phi}_2 = g_2$. In fact, $\phi \in C^{(\alpha)}$ for $\alpha < M-1$ since

$$\int_{\mathbb{R}} |\hat{\phi}(\omega)|(1+|\omega|)^\alpha d\omega < \infty.$$

Note that $g_1 = \prod_{k=1}^{\infty}(\frac{1+z_k}{2})^M$ so ϕ_1 is the cardinal B-spline of order $M-1$. $\square$.

Our next result provides insight as to how to construct $\boldsymbol{\phi}$ iteratively.

Theorem 2.3 *Let $P(\omega)$ satisfy the hypotheses of Theorem 2.1. Suppose there exists $f \in L^2(\mathbb{R})$ such that*

$$\hat{f}(0) = 1,$$

and

$$\hat{f} \textit{ is continuous at } 0.$$

Define $\boldsymbol{\phi}_0(\omega) = \hat{f}(\omega)u$ and assume there exists $C > 0, \eta > 1$ such that

$$\left|\prod_{k=1}^{n} P_k \hat{\boldsymbol{\phi}}_0(\frac{\omega}{2^n})\right| \leq \frac{C}{(1+|\omega|)^\eta} \tag{15}$$

for all n, ω. Then for all ω

$$\lim_{n\to\infty} \boldsymbol{\phi}_n(\omega) = \boldsymbol{\phi}(\omega) \tag{16}$$

where the convergence is uniform and

$$\boldsymbol{\phi}_n(x) = \sum_{k=0}^{N} \mathbf{C_k}\boldsymbol{\phi}_{n-1}(2x-k). \tag{17}$$

Proof. The Fourier transform formulation of (17) is

$$\hat{\boldsymbol{\phi}}_n(\omega) = P(\frac{\omega}{2})\hat{\boldsymbol{\phi}}_{n-1}(\frac{\omega}{2}).$$

Iterating this equation $n-1$ times, we have

$$\hat{\boldsymbol{\phi}}_n(\omega) = \prod_{k=1}^{n} P(\frac{\omega}{2^k})\hat{\boldsymbol{\phi}}_0(\frac{\omega}{2^n}) = \hat{f}(\frac{\omega}{2^n})\prod_{k=1}^{n} P(\frac{\omega}{2^k})u. \tag{18}$$

By the continuity of $\hat{f}$,

$$\lim_{n\to\infty} \hat{\boldsymbol{\phi}}_n(\omega) = \hat{f}(0)\prod_{k=1}^{\infty} P_k u = g(\omega)$$

for each ω, and $|g(\omega)| \leq \frac{C}{(1+|\omega|)^\eta}$ for all ω. Hence $g \in L^2(\mathrm{I\!R}) \cap L^1(\mathrm{I\!R})$ and there exists $\phi \in L^2(\mathrm{I\!R})$ such that $\hat{\phi} = g$.

The inequality (15) guarantees that $\hat{\phi}_n \in L^1(\mathrm{I\!R})$ for all n, so by the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \|\hat{\phi}_n - \hat{\phi}\|_{L^1} = 0.$$

Finally,

$$|\phi_n(x) - \phi(x)| \leq \frac{1}{2\pi}\|\hat{\phi}_n - \hat{\phi}\|_{L^1} \quad \text{a.e.},$$

which proves (16). $\square$

We have an immediate corollary concerning the compact support of ϕ.

Corollary 2.4 *If ϕ_0 has compact support, then ϕ has compact support.*

3 Characterizing Scaling Vectors and their Support.

We motivate the main result of the section with an example given in [9]. There, spline wavelets of multiplicity r for $L^2(\mathrm{I\!R})$ were constructed. An application of their results with $V_0 = S_3^1(\mathbb{Z})$, the space of all piecewise continuously differientable cubic polynomials with possible breakpoints at the integers, shows that two B-spline scaling functions $B_1(x) := B(x|0,0,1,1,2)$ and $B_2(x) := B(x|0,1,1,2,2)$ pictured in Figure 3.1 below are needed along with their integer translates to form a basis for V_0. (In defining B_1 and B_2, we have displayed the knot sequence. For more details, please see [1].)

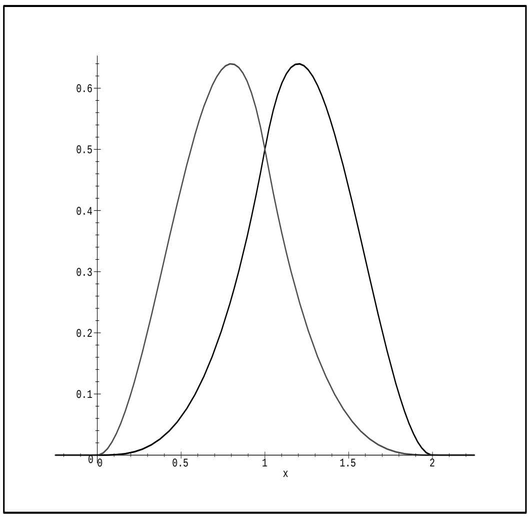

Figure 3.1.

Clearly, these functions are positive on their support, and for $|k_1 - k_2| < 3$, $B_1(x - k_1)$ and $B_2(x - k_2)$ are not orthogonal. In applications, it is desirable to construct generators for $S_3^1(\mathbb{Z})$ that possess more orthogonality. Unfortunately, not all linear combinations of scaling functions generate a Riesz basis for V_0. (See [7], pps. 177-178).

Suppose $B_{\boldsymbol{\phi}}$ generates a Riesz for V_0. We want to characterize those linear combinations of ϕ_i that also generate a Riesz basis for the same V_0. The main result of this section gives insight as to how we characterize such scaling functions.

To this end, a few basic ideas are needed. Let us define the *support* of the vector $\boldsymbol{\phi}$ by

$$\text{supp}\,(\boldsymbol{\phi}) \ = \cup_{j=1}^{r} \text{supp}\,(\phi_j) \ . \tag{19}$$

Suppose that $B_{\boldsymbol{\phi}}$ is a basis for V_0 and $\boldsymbol{\phi}$ has support length M_ϕ. Then $\boldsymbol{\phi}$ is said to have *minimal support* if for any other vector $\mathbf{g}$ with support length M_g, $B_{\mathbf{g}}$ a basis for V_0, it follows that $M_\phi \leq M_g$.

Following Goodman and Lee [9], we say a scaling vector $\boldsymbol{\phi}$ and its integer translates ϕ_k are *locally linearly independent* on a nontrivial interval (a, b) if whenever

$$\sum_k c_k \phi_k \equiv 0$$

on (a, b) then $c_k = 0$ for all k for which ϕ_k is not identically zero on (a, b).

Let $\mathcal{S}$ be the set of all compactly scaling vectors that generate the same MRA of $L^2(\mathbb{R})$. Let L_ϕ (R_ϕ) denote the left (right) endpoint of supp ϕ. By using integer translates in V_0, we may assume that

$$0 \leq L_\phi < 1$$

for all $\phi \in \mathcal{S}$. In the case of a single scaling function, it follows that $L_\phi = 0$. (See, for instance [2], Section 5.2.) However, as we shall see in Example 3.6, L_ϕ and R_ϕ need not be integers with $r > 1$ scaling functions.

Theorem 3.1 *Let $\boldsymbol{\phi}$ have compact support and assume that $\boldsymbol{\phi}$ satisfies the matrix refinement equation*

$$\boldsymbol{\phi}(x) = \sum_{k=0}^{N} \mathbf{C_k} \boldsymbol{\phi}(2x - k), \tag{20}$$

where $\mathbf{C_k} \in \mathbb{R}^{r \times r}$. Furthermore, suppose that

(i) $\{\phi_1, \ldots, \phi_r\} \subseteq V_0$ and its integer translates are locally linearly independent on every nontrivial interval;

(ii) $\{v_1, \ldots, v_q\} \subseteq V_0$ and its integer translates are locally linearly independent on every nontrivial interval;

(iii) supp (ϕ_i) = supp (ϕ_j) , $i, j = 1, \ldots r$, and supp (v_i) = supp (v_j) , $i, j = 1, \ldots, q$.

Then

$$r = q \text{ and } \mathbf{v} = A\boldsymbol{\phi} \text{ for some nonsingular } A \in \mathbb{R}^{r\times r} \tag{21}$$

$$\Longleftrightarrow$$

1) $V_0 = \overline{span\{v_j(x-k), j = 1, \ldots, q, k \in \mathbb{Z}\}}^{L^2}$

2) $\mathbf{v}$ *satisfies a matrix refinement equation.*

3) $supp\,(\mathbf{v}) \ = supp\,(\boldsymbol{\phi})$

Proof. ($\Leftarrow$) Without loss of generality, suppose that $\mathrm{supp}\,(\mathbf{v}) \ = \mathrm{supp}\,(\boldsymbol{\phi}) \ = [L_\phi, R_\phi]$, where

$$0 \le L_\phi < 1 \quad \text{and} \quad M-1 \le R_\phi < M,$$

for some positive integer M. Since $\{v_1, \ldots v_q\} \subseteq V_0$, there exist $q \times r$ matrices B_j such that

$$\mathbf{v}(x) = \sum_{j\in\mathbb{Z}} B_j \boldsymbol{\phi}(x-j). \tag{22}$$

Let $D = \mathbb{Z} \cap \{(-\infty, -M-1] \cup [0, \infty)\}$ and fix $n \in D$. For any $x \in (M+n, M+n+1)$, we have

$$\mathbf{v}(x) = \sum_{j=n+1}^{n+M} B_j \boldsymbol{\phi}(x-j) \equiv 0,$$

since $\mathrm{supp}(\mathbf{v}) = \mathrm{supp}(\boldsymbol{\phi})$. The locally linear independence of $\{\phi_1, \ldots, \phi_r\}$ and its integer translates and hypothesis (iii) imply that

$$B_j = 0, \quad \text{for } j = n+1, \ldots, n+M.$$

The definition of D yields $B_j = 0$, for $j \ne 0$, that is,

$$\mathbf{v}(x) = B_0 \boldsymbol{\phi}(x), \qquad \text{for all } x \in \mathbb{R}.$$

Since the $\{v_i\}$ are locally linearly independent, there is no $1 \times q$ vector $y \ne 0$ such that $y\mathbf{v}(x) = yB_0\boldsymbol{\phi}(x) = 0$. Hence, B_0 must be of full rank and $q \le r$. Reversing the roles of $\mathbf{v}$ and $\boldsymbol{\phi}$ and using a similar argument, we can show that there exists a full rank matrix $A_0 \in \mathbb{R}^{r\times q}$ such that $\boldsymbol{\phi}(x) = A_0\mathbf{v}(x)$ and $r \le q$. Thus $r = q$ and $A_0 = B_0^{-1}$.

($\Rightarrow$) 1) follows since $\boldsymbol{\phi} = A^{-1}\mathbf{v}$. 2) can be shown using the matrix refinement equation (20):

$$\mathbf{v}(x) = A\boldsymbol{\phi}(x) = A\sum_{k=0}^{N} \mathbf{C_k}(A^{-1}\mathbf{v})(2x-k)$$

so that

$$\mathbf{v}(x) = \sum_{k=0}^{N} (A\mathbf{C_k}A^{-1})\mathbf{v}(2x-k).$$

3) follows from that fact that since each v_j is a linear combination of the components of ϕ, $\operatorname{supp}(\mathbf{v}) \subseteq \operatorname{supp}(\phi)$. Using a similar argument, we have $\operatorname{supp}(\phi) \subset \operatorname{supp}(\mathbf{v})$. □

Remark. It was proven in [10], Theorem 5.3, that if ϕ is compactly supported and has linearly independent integer translates then ϕ generates a Riesz basis of $L^2(\mathbb{R})$. It follows that if ϕ satisfies the hypotheses of Theorem 3.1 and $A \in \mathbb{R}^{r\times r}$ is nonsingular, then $\mathbf{v} = A\phi$ will generate a Riesz basis of $L^2(\mathbb{R})$.

Before stating a corollary of Theorem 3.1, we give an example that illustrates the results we have reported in this section.

Example 3.2 *Consider the B-spline scaling vector of Lawton, Lee, and Shen, [12], and Goodman and Lee [9] for the space* $V_0 = S_3^1(\mathbb{Z})$.

The two scaling functions needed to generate $V_0 = S_3^1(\mathbb{Z})$ are pictured in Figure 3.1. We seek generators of V_0 that are orthogonal to each other. Using the results of Theorem 3.1, we know that the locally linearly independent Hermite interpolants $H_1(x)$ and $H_2(x)$ pictured in Figure 3.2 indeed span V_0, solve a matrix refinement equation, and are supported on $[0,2]$ since

$$\begin{bmatrix} H_1(x) \\ H_2(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1/3 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} B_1(x) \\ B_2(x) \end{bmatrix}.$$

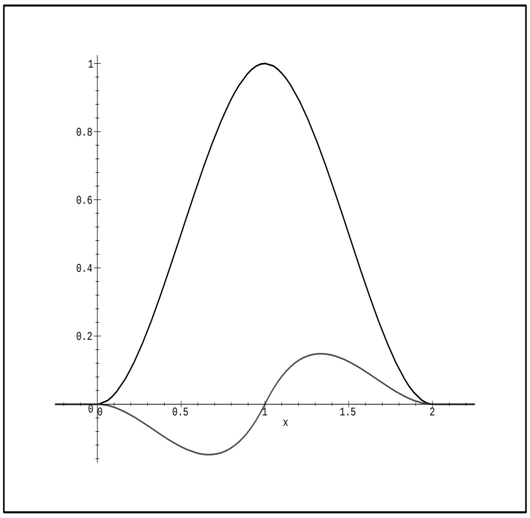

Figure 3.2.

Moreover, the matrix refinement equation satisfied by $B_1(x)$ and $B_2(x)$ is

$$\mathbf{B}(x) = \begin{bmatrix} 1/4 & 5/8 \\ 0 & 1/8 \end{bmatrix} \mathbf{B}(2x) + \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \mathbf{B}(2x-1) + \begin{bmatrix} 1/8 & 0 \\ 5/8 & 1/4 \end{bmatrix} \mathbf{B}(2x-2).$$

The scaling vector

$$\mathbf{H}(x) = \begin{bmatrix} H_1(x) \\ H_2(x) \end{bmatrix}$$

may be useful in applications since the scaling functions are orthogonal and possess symmetry/antisymmetry properties. In addition, since Goodman and Lee [9] have shown that $B_1(x)$ and $B_2(x)$ generate a Riesz basis for V_0 so that we may employ the remark following Theorem 3.1 to prove that $\mathbf{H}$ also generates a Riesz basis for V_0.

The following corollary resulting from Theorem 3.1 characterizes those scaling vectors for V_0 that possess minimal support.

Corollary 3.3 *Suppose that $\boldsymbol{\phi}, \boldsymbol{\phi}^* \in \mathcal{S}$ satisfy the hypotheses of Theorem 3.1 and have minimal support. Then*

$$supp(\boldsymbol{\phi}) = supp(\boldsymbol{\phi}^*) \quad \textit{and} \quad \boldsymbol{\phi} = A\boldsymbol{\phi}^*$$

for some nonsingular $A \in \mathbb{R}^{r\times r}$.

Proof. We need only show that $R_\phi = R_{\phi^*}$, for then $\text{supp}(\boldsymbol{\phi}) = \text{supp}(\boldsymbol{\phi}^*)$ since $\boldsymbol{\phi}, \boldsymbol{\phi}^*$ have minimal support. The second conclusion then follows immediately from Theorem 3.1. From the discussion above Theorem 3.1 and the minimal support of $\boldsymbol{\phi}, \boldsymbol{\phi}^*$, there is some integer M for which

$$0 \le L_\phi \le L_{\phi^*} < 1 \quad \text{and} \quad M - 1 \le R_\phi \le R_{\phi^*} < M.$$

Since $\boldsymbol{\phi}^* \in V_0$,

$$\boldsymbol{\phi}^*(x) = \sum_{j\in\mathbb{Z}} B_j \boldsymbol{\phi}(x - j)$$

for some matrices B_j.

Using the argument given in the proof of Theorem 3.1, local linear independence, and hypothesis (iii) in Theorem 3.1, we conclude that $B_j = 0$ for $j \neq 0$, so

$$\boldsymbol{\phi}^*(x) = B_0\boldsymbol{\phi}(x).$$

Now if $R_\phi < R_{\phi^*}$ then this equation becomes $\phi^*(x) = B_0 \cdot 0 = 0$ on (R_ϕ, R_{ϕ^*}), which is impossible by the definition of R_{ϕ^*}. Hence $R_\phi = R_{\phi^*}$ $\square$.

Note that this corollary is a generalization of the single scaling function case (see [2]) where minimally supported scaling functions are unique up to a normalization factor.

We conclude this section with an example that illustrates the necessity of the equal support condition (iii) in Theorem 3.1.

Example 3.4 *Consider the wavelets constructed from fractal scaling functions $\phi_1(x)$, $\phi_2(x)$ in [8] that satisfy the matrix refinement equation*

$$\begin{aligned}\boldsymbol{\phi}(x) \quad = \quad & \begin{bmatrix} 3/5 & \frac{4\sqrt{2}}{5} \\ \frac{-1}{10\sqrt{2}} & -3/10 \end{bmatrix} \boldsymbol{\phi}(2x) + \begin{bmatrix} 3/5 & 0 \\ \frac{9}{10\sqrt{2}} & 1 \end{bmatrix} \boldsymbol{\phi}(2x-1) + \\ & \begin{bmatrix} 0 & 0 \\ \frac{9}{10\sqrt{2}} & -3/10 \end{bmatrix} \boldsymbol{\phi}(2x-2) + \begin{bmatrix} 0 & 0 \\ \frac{-1}{10\sqrt{2}} & 0 \end{bmatrix} \boldsymbol{\phi}(2x-3). \end{aligned} \tag{23}$$

The orthogonal scaling functions ϕ_1 and ϕ_2 are shown in Figure 3.3 below.

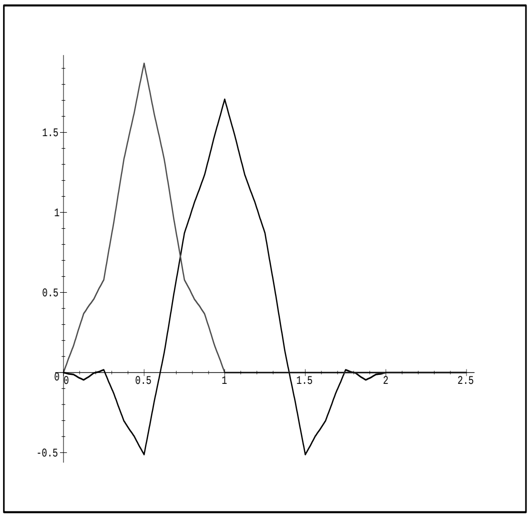

Figure 3.3

Note that supp(ϕ_1) $= [0,2] \neq$ supp(ϕ_2) $= [0,1]$. Clearly, if ϕ_1 and ϕ_2 generate a Riesz basis for V_0, then so does $\bar{\phi_1}(x) = \phi_1(x)$ and $\bar{\phi_2}(x) = \phi_2(x-1)$. However, there exists no invertible matrix $A \in \mathbb{R}^{2\times 2}$ such that

$$\begin{bmatrix} \bar{\phi_1} \\ \bar{\phi_2} \end{bmatrix} = A \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

It is known that in the single scaling function setting, the number of nonzero coefficients in the refinement equation is related to the support length of the scaling functions [2, 7]. This relationship is more complex in the multiple scaling function case, due to the existence of nonzero nilpotent elements of $\mathbb{R}^{r\times r}$ for $r > 1$.

Theorem 3.5 *Assume that $\boldsymbol{\phi}$ has compact support, is locally linearly independent, and satisfies the hypotheses of Theorem 2.3. Furthermore, let $\boldsymbol{\phi}(x) = \sum_{k=0}^{N} \mathbf{C_k}\boldsymbol{\phi}(2x-k)$, $\mathbf{C}_0, \mathbf{C}_N \neq 0$, $\mathbf{C_k} \in \mathbb{R}^{r\times r}$.*

1) If $\mathbf{C}_N$ is nilpotent, then supp$(\boldsymbol{\phi})$ $\subseteq [0, N - \frac{1}{2^r-1}]$.

2) If $\mathbf{C}_0$ *is nilpotent, then supp* $(\boldsymbol{\phi}) \subseteq [\frac{1}{2^r-1}, N]$.

3) If neither $\mathbf{C}_0$ *nor* $\mathbf{C}_N$ *is nilpotent, then supp* $(\boldsymbol{\phi}) = [0, N]$.

Proof. Assume that $\mathbf{C}_N$ is nilpotent. The proof for $\mathbf{C}_0$ nilpotent follows analogously. Let A be the similarity matrix that puts $\mathbf{C}_N$ into (real) Jordan form, and let $\tilde{\mathbf{C}}_k = A^{-1}\mathbf{C_k}A$, and $\tilde{\boldsymbol{\phi}} = A^{-1}\boldsymbol{\phi}$ for $k = 1, \ldots r$. Then

$$\tilde{\boldsymbol{\phi}}(x) = \sum_{k=0}^{N} \tilde{\mathbf{C}}_k \tilde{\boldsymbol{\phi}}(2x - k). \tag{24}$$

Let E_i be the right endpoint of the support of $\tilde{\boldsymbol{\phi}}_i$ for $i = 1, \ldots, r$. Using the recursion formula from Theorem 2.3, we observe that $E_i \leq N$ for all i. Since $\tilde{\mathbf{C}}_N$ is lower triangular with 0's on the diagonal, the first row of the matrix refinement equation (24) guarantees that

$$2E_1 - (N-1) \leq N, \quad \text{or} \quad E_1 \leq N - \frac{1}{2}.$$

The second row of (24) consequently yields

$$2E_2 - N \leq N - \frac{1}{2}, \quad \text{or} \quad E_2 \leq N - \frac{1}{4}.$$

Proceeding in this manner, we obtain

$$E_i \leq N - 2^{-i}, i = 1, \ldots, r. \tag{25}$$

Returning to the first row of (24), we are assured by (25) and the lower triangular form of $\tilde{\mathbf{C}}_N$ that

$$2E_1 - (N-1) \leq N - \frac{1}{2^r}, \quad \text{or} \quad E_1 \leq N - \frac{2^r + 1}{2^{r+1}}.$$

Proceeding as before down the rows of (24), we obtain

$$E_i \leq N - \frac{2^r + 1}{2^{r+i}} \tag{26}$$

for $i = 1, \ldots, r$, or supp $\left(\tilde{\boldsymbol{\phi}}\right) \subseteq \left[0, N - \frac{2^r+1}{2^{r+1}}\right]$.

Repeating this routine a total of K times, we have

$$\text{supp}\left(\tilde{\boldsymbol{\phi}}\right) \subseteq \left[0, N - \frac{\sum_{j=0}^{K-1} 2^{jr}}{2^{rK}}\right]. \tag{27}$$

Passing to the limit yields

$$\text{supp}\left(\tilde{\boldsymbol{\phi}}\right) \subseteq \left[0, N - \frac{1}{2^r - 1}\right].$$

Note that $\tilde{\boldsymbol{\phi}}(x) = 0$ if and only if $\boldsymbol{\phi}(x) = A\tilde{\boldsymbol{\phi}}(x) = 0$, so

$$\mathrm{supp}\,(\boldsymbol{\phi}) \;\subseteq \left[0, N - \frac{1}{2^r - 1}\right].$$

To prove part 3), again let A be a similarity matrix that puts $\mathbf{C}_N$ into(real) Jordan form, with nonzero eigenvalues of $\tilde{\mathbf{C}}_N = A^{-1}\mathbf{C}_N A$ in the upperleft hand corner. That is

$$|(\tilde{\mathbf{C}})_{ii}| \geq |(\tilde{\mathbf{C}})_{i+1,i+1}|$$

for $i = 1, 2, \ldots, r-1$.

Define $\tilde{\boldsymbol{\phi}}$ and E_i as in the proof of part 1), and let $\mathrm{supp}\,(\phi_1) \;= [a, E_1]$, where clearly $0 \leq a \leq E_1 \leq N$ by the recursion formula in Theorem 2.3.

Now $\mathrm{supp}\,(\phi_1(2x - N)) \;= [\frac{N+a}{2}, \frac{N+E_1}{2}]$, so the form of $\tilde{\mathbf{C}}_N$ forces

$$\mathrm{supp}\,\Big((\tilde{\mathbf{C}}_N \phi(2x - N))_1\Big) \;= \left[\frac{N+a}{2}, \frac{N+E_1}{2}\right]. \tag{28}$$

This fact, the refinement equation (24) and $\mathrm{supp}\,\sum_{k=0}^N \tilde{\mathbf{C}}_k \tilde{\boldsymbol{\phi}}(2x-k) \subseteq \cup_{k=0}^N \mathrm{supp}\,\Big(\tilde{\boldsymbol{\phi}}(2x-k)\Big)$ yield

$$E_1 \leq \frac{N + E_1}{2}.$$

By local linear independence this inequality becomes an equality; whence $E_1 = N$. Therefore

$$[a, N] \subseteq \mathrm{supp}(\tilde{\boldsymbol{\phi}}).$$

As $\tilde{\boldsymbol{\phi}}(x) = 0$ if and only if $\boldsymbol{\phi}(x) = A\tilde{\boldsymbol{\phi}}(x) = 0$, we have

$$[a, N] \subseteq \mathrm{supp}(\boldsymbol{\phi}).$$

An analogous argument with $\mathbf{C}_0$ yields $\mathrm{supp}\boldsymbol{\phi} \subseteq [0, b]$ for some $0 \leq b \leq N$, so we can conclude that

$$\mathrm{supp}\,(\boldsymbol{\phi}) \;= [0, N]. \;\Box$$

Note that in Examples 2.2 and 3.4, the $\mathbf{C}_N$ matrices are nilpotent and the support length of $\boldsymbol{\phi}$ is stricly less than $N - \frac{1}{2^r} - 1$. We conclude the section with an example that illustrates the inclusion in 1) can be an equality.

Example 3.6 *Scaling vectors illustrating the support properties of Theorem 3.5 can be generated using the $r \times r$ symbols (with $z = e^{-i\omega/2}$)*

$$P(\omega) = \left(\frac{1+z}{2}\right)^M \begin{bmatrix} 1/r & 1/r & \cdots & 1/r & 1/r \\ z/r & 1/r & \cdots & 1/r & 1/r \\ 1/r & z/r & \cdots & 1/r & 1/r \\ \vdots & \ddots & \ddots & \ddots & 1/r \\ 1/r & \cdots & 1/r & z/r & 1/r \end{bmatrix},$$

for $M \geq 1$.

The eigenvector of $P(0)$ associated with eigenvalue 1 is $u = (1,1,\ldots,1)^T$, and a routine calculation shows that

$$\begin{aligned} |g_i(\omega)| &\leq \|g(\omega)\|_{L^2} \\ &= \|\lim_{n\to\infty}\prod_{k=1}^{n} P_k u\|_{L^2} \\ &\leq \left|\prod_{k=1}^{\infty}\left(\frac{1+e^{-i\omega/2^k}}{2}\right)^M\right| \\ &\leq \frac{C}{(1+|\omega|)^M} \end{aligned}$$

so each $\phi_i \in L^2(\mathbb{R})$, and in fact $\phi_i \in C^{(M-2)}$. Note that P_{M+1} is nilpotent. If $M \geq 2$ and $\boldsymbol{\phi}_0 = \begin{bmatrix} N_2(x) \\ N_2(x) \end{bmatrix}$, where $N_2(x)$ is the cardinal B-spline of order 2, then the hypotheses of Theorem 3.5 are satisfied and supp $(\boldsymbol{\phi})$ is exactly $[0, M+1-\frac{1}{2^r-1}]$.

The scaling functions generated when $r = M = 2$ are displayed in Figure 3.4 below.

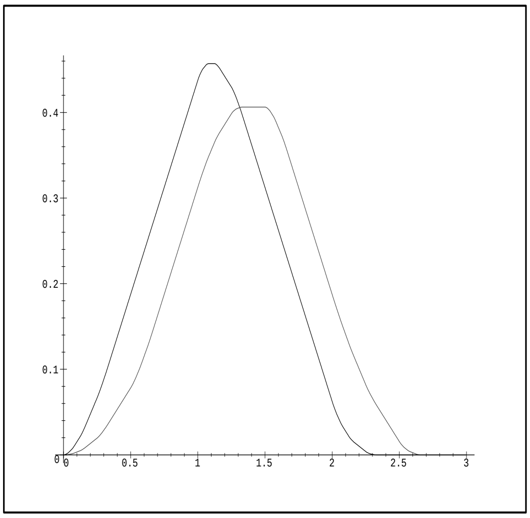

Figure 3.4

Remark. After the completion of this paper, the authors learned of work by A. Cohen, I. Daubechies, and G. Plonka ("Regularity of Refinable Function Vectors") that overlaps with some of the material in Section 2.

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