Moment Computation in Shift Invariant Spaces

David A. Eubanks* Patrick J. Van Fleet† Jianzhong Wang‡

Abstract

An algorithm is given for the computation of moments of $f \in S$, where S is either a principal h-shift invariant space or S is a finitely generated h-shift invariant space. An error estimate for the rate of convergence of our scheme is also presented. In so doing, we obtain a result for computing inner products in these spaces. As corollaries, we derive Marsden-type identities for principal h-shift invariant spaces and finitely generated h-shift invariant spaces. Applications to wavelet/multiwavelet spaces are presented.

AMS(MOS) 1991 subject classification: 41A25.
Keywords and Phrases: (Principal) h-shift invariant space, finitely generated h-shift invariant space, moment, Marsden's identity, wavelet, multiwavelet.

---

1 Introduction

We consider the computation of the moment $m_\beta(f)$ of a function $f \in L^2(\mathbb{R})$. To this end, we project f into either an h-principal invariant subspace or a finitely generated shift invariant subspace. The approximation order and other characteristics of such spaces have been studied extensively in the fundamental paper [1] and again in [11, 15, 10].

Our main result deals with the computation of inner products in shift invariant spaces. The advantage of utilizing these spaces is the fact that we can often construct stable bases whose elements are integer translates of a compactly supported function or finite compactly supported functions. Thus the computation of the inner product is easily implemented on a computer. As corollaries to our main result, we obtain as a special case the ability to compute moments of functions $f_0 \in V_0$. We then show how the process can be refined to obtain

*Dept. of Mathematics, Coker College, Hartsville, SC 29550 USA, eubanks@coker.edu.

†Department of Mathematics, Sam Houston State University, Huntsville, TX 77341 USA, vanfleet@guy.shsu.edu, research partially supported by National Science Grant DMS9503282 and The Texas Regional Institute for Environmental Studies.

‡Department of Mathematics, Sam Houston State University, Huntsville, TX 77341 USA, jwang@galois.shsu.edu, research partially supported by National Science Foundation Grant DMS9503282

moments of the function $f_h \in V_h$. The idea is to construct a sequence of shift invariant spaces V_h approximating $L^2(\mathbb{R})$ in hopes of eventually approximating the moment of $f \in L^2(\mathbb{R})$ by the moment of $f_h \in V_h$. In the case where the vector that generates the finitely generated shift invariant space V_0 is refinable, we use a result of Cohen, Daubechies, and Plonka [5] in order to obtain an estimate of the error $|m_\beta(f) - m_\beta(f_h)|$.

As a consequence of our main result, we characterize a Marsden's identity for finitely generated shift invariant spaces. Recall Marsden's identity gives the explicit representation of x^n in terms of B-splines [12]. A multivariate analog for box splines is given in [3].

The outline of this paper is as follows: In Section 2, we give basic definitions and elementary results necessary to the sequel. An inner product theorem and related corollaries concerning moment computation and Marsden's identity are given in Section 3. In addition, we give an error estimate for the difference between the moment of $f \in L^2(\mathbb{R})$ and the moment of its orthogonal projection f_h in a shift invariant subspace of $L^2(\mathbb{R})$. The final section contains moment recursion formulas for refinable functions and vectors as well as examples of illustrating our results.

2 Notation, Definitions, and Basic Results

In this section we introduce notation, definitions, and basic results used throughout the remainder of the paper. Let us begin by defining various types of shift invariant spaces.

Suppose V_h is a linear space and $h > 0$. Then V_h is said to be an *h-shift invariant space* if

$$f \in V_h \Longrightarrow f(\cdot - h) \in V_h.$$

V_h is a *principal h-shift invariant space* if V_h is an h-shift invariant space generated by a compactly supported function $\phi \in V_h$. That is

$$f \in V_h \Longrightarrow f(x) = \sum_{k \in \mathbb{Z}} c_k \phi(x - hk).$$

When $h = 1$, we will suppress the h in the definitions above and refer to the spaces as shift invariant and principal shift invariant, respectively. As a matter of convention, we will denote an h-shift invariant space generated by ϕ by $V_h(\phi)$. We will also be interested in shift invariant spaces generated by several functions. That is, we define a *finitely generated shift invariant space* V_h by insisting that

$$f \in V_h \Longrightarrow f(x) = \sum_{k \in \mathbb{Z}} (\boldsymbol{a}^k)^T \boldsymbol{\phi}(x - hk),$$

where now

$$\boldsymbol{\phi}(x) = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}(x),$$

and the $\boldsymbol{a}^k \in \mathbb{R}^r$. Such a space generated by $\boldsymbol{\phi}$ will be denoted by $V_h(\boldsymbol{\phi})$.

We readily observe the following properties:

(1) If V is a shift invariant space, then $V_h = \{f(\cdot) : f(h\cdot) \in V\}$ is an h-shift invariant space.

(2) If V is a shift invariant space generated by $\boldsymbol{\phi}$, then V_h is an h-shift invariant space generated by $\boldsymbol{\phi}(\frac{\cdot}{h})$.

We will say that the h-shift invariant space V_h is of *degree* n and write $\deg(V_h) = n$ if $x^k \in V_h$, $k = 0, \ldots, n$, but $x^{n+1} \notin V_h$. Since the degree of a polynomial is invariant under dilation, we observe that $\deg(V_h) = \deg(V)$.

If $\deg(V(\phi)) = n$, we will call any identity of the form

$$x^\ell = \sum_{k\in\mathbb{Z}} c_k^\ell \phi(x-k), \qquad \ell = 0, \ldots, n$$

a *Marsden's identity.* The identity is easily generalized to the case where V is generated by $\boldsymbol{\phi}$.

Suppose ϕ generates the principal shift invariant space V. In this paper, we always assume that ϕ is integrable. Recall that if ϕ is compactly supported and integrable, then ϕ is also in L^2. Now we say a function $\phi^* \in V$ is the *dual* of ϕ if

$$<\phi^*(\cdot - k), \phi(\cdot - j)>= \delta_{kj},$$

where $<f,g>= \int_{\mathbb{R}} f(x)g(x)dx$ is the standard inner product and

$$\delta_{kj} = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{otherwise.} \end{cases}$$

Analogously, if vector $\boldsymbol{\phi}$ generates the space V, then we will define its dual $\boldsymbol{\phi}^* \in V$ as the vector of functions that satify

$$<\boldsymbol{\phi}_\ell^*(\cdot - k), \boldsymbol{\phi}_m(\cdot - j)>= \delta_{\ell m}\delta_{kj}.$$

We will say ϕ is *stable* if

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(\omega + 2k\pi)|^2 > 0 \qquad \forall \omega \in \mathbb{R}$$

and vector $\boldsymbol{\phi}$ is stable if the $r \times r$ matrix $\hat{\boldsymbol{\Phi}} := (\hat{\Phi}_{ij}(\omega))_{i,j=1}^r$ is positive definite where

$$\hat{\Phi}_{ij}(\omega) = \sum_{k\in\mathbb{Z}} \hat{\Phi}_i(\omega + 2k\pi)\overline{\hat{\Phi}_j(\omega + 2k\pi)}.$$

Recall that if $\boldsymbol{\phi}$ is stable, then $\boldsymbol{\phi}^*$ also generates a stable basis for V.

Suppose $\{\boldsymbol{\phi}_\ell(\cdot - k)\}$, $k \in \mathbb{Z}$, $\ell = 1, \ldots, r$ forms a basis for V. Then it also forms a basis for $V \cap L^2(\mathbb{R})$. For convenience, we will use V to denote $V \cap L^2(\mathbb{R})$. We say that V_h provides L^2-approximation order m if, for each sufficiently smooth function $f \in L^2(\mathbb{R})$,

$$||f - \text{Proj}_V f||_2 \leq C h^m.$$

When $\boldsymbol{\phi}$ is a compactly supported vector in L^2, then $V(\boldsymbol{\phi})$ provides L^2-approximation order m if and only if $V(\boldsymbol{\phi})$ contains Π_{m-1}, the set of all polynomials of degree $\leq m-1$ [11].

Remark. If $\boldsymbol{\phi}$ is refinable, i.e.,

$$\boldsymbol{\phi}(x) = \sum_{k=0}^{N} P_k \boldsymbol{\phi}(2x - k) \tag{1}$$

where the P_k are $r \times r$ matrices, then $\Pi_{m-1} \subset V(\boldsymbol{\phi})$ is equivalent to the existence of solutions to two systems of equations involving the refinement mask of $\boldsymbol{\phi}$ (see [5, 10, 15]).

When $\boldsymbol{\phi}$ is refinable in the sense of (1), we can obtain estimates on the accuracy of approximating moments of $f \in L^2(\mathbb{R})$ using projections of f into $V_h(\boldsymbol{\phi})$.

Proposition 2.1 *Suppose $\boldsymbol{\phi} \in \mathbb{R}$ is a refinable vector and assume $\{\phi_j(\cdot - k) : k \in \mathbb{Z}, j = 1, \ldots r\}$ forms a linearly independent basis for the space $V(\boldsymbol{\phi})$ with deg$(V(\boldsymbol{\phi})) = n-1$. Further suppose that $f \in L^2(\mathbb{R})$ is compactly supported and sufficiently smooth. We have:*

$$|m_\beta(f) - m_\beta(Proj_{V_h} f)| \leq C_f h^n \tag{2}$$

where $\beta = 0, \ldots, n-1$ and C_f is a constant that depends on the support width of f.

Proof. The result of [5] guarantees that V provides approximation order n. If S denotes the support of f, we have

$$\begin{aligned} |m_\beta(f) - m_\beta(\text{Proj}_{V_m} f)| &= |< x^\beta \chi_S(x), (f - \text{Proj}_{V_m} f) > | \\ &\leq ||x^\beta \chi_S(x)|| \cdot ||f - \text{Proj}_{V_m} f|| \\ &\leq C_f h^m. \qquad \square \end{aligned}$$

3 Main Results

The main goal of this paper is to provide an algorithm for computing moments in principal (or finitely generated) shift invariant spaces. Once the algorithm is in place, we will use it to attempt to approximate the moment $m_\beta(f)$ of $f \in L^2(\mathbb{R})$. In order to obtain formula for computing moments of $f_0 = \text{Proj}_V f$ and subsequent moments in refined spaces, we establish the following result.

Theorem 3.1 *Suppose $\boldsymbol{\phi}$ is compactly supported, stable and that $\boldsymbol{\phi}^*$ is its dual. Denote by $V(\boldsymbol{\phi})$ the space generated by $\boldsymbol{\phi}$. Assume $f \in V(\boldsymbol{\phi})$ satisfies the decay condition*

$$|f(x)| \le \frac{C}{1+|x|^\alpha}, \qquad \alpha > 1, \tag{3}$$

and $g \in V(\boldsymbol{\phi})$ satisfies the growth condition

$$|g(x)| \le C(1+|x|^\beta), \qquad \alpha - \beta > 1 \tag{4}$$

where C is an absolute constant. Let

$$f(x) = \sum_{k\in\mathbb{Z}} (\boldsymbol{a}^k)^T \boldsymbol{\phi}(x-k), \qquad g(x) = \sum_{k\in\mathbb{Z}} (\boldsymbol{b}^k)^T \boldsymbol{\phi}^*(x-k). \tag{5}$$

Then $< f, g >= \sum_{k\in\mathbb{Z}} (\boldsymbol{a}^k)^T \boldsymbol{b}^k$.

Proof. Without loss of generality, assume that $\text{supp}(\boldsymbol{\phi}) = [0, L]$. (By the support of a vector $\boldsymbol{\phi} \in \mathbb{R}^r$, we mean $\text{supp}(\boldsymbol{\phi}) = \cup_{k=0}^r \text{supp}(\phi_k)$). First note that since $\boldsymbol{\phi}$ is compactly supported, $\boldsymbol{\phi}^*$ is of exponential decay. That is, for $k = 1, \dots, r$:

$$|\phi_k^*(x)| \le C_1 e^{-\gamma|x|} \tag{6}$$

for some $\gamma > 0$.

We shall now estimate the decay rate of a_i^k and b_i^k, $k \in \mathbb{Z}$, and $i = 1, \dots r$. First we show that for sufficiently large k and constant $C_2 > 0$, we have

$$|a_i^k| \le C_2 |k|^{-\alpha}. \tag{7}$$

Since $\boldsymbol{\phi}$ is compactly supported and stable, the set $\{\boldsymbol{\phi}^*(\cdot - k)\}_{k\in\mathbb{Z}}$ forms an unconditional basis for $V(\boldsymbol{\phi})$. Therefore,

$$a_i^k = \int_{\mathbb{R}} f(x) \boldsymbol{\phi}_i^*(x-k).$$

Then for sufficiently large k, we have

$$\begin{aligned} |a_i^k| &= |\int_{\mathbb{R}} f(x+k)\phi_i^*(x)dx| \\ &\le C_1 \left(\int_{|x|\le\frac{\alpha}{\gamma}\ln|k|} \frac{dx}{1+|x+k|^\alpha} + \int_{|x|>\frac{\alpha}{\gamma}\ln|k|} e^{-\gamma x} dx \right) \\ &\le C_2 |k|^{-\alpha} \end{aligned}$$

Using an analogous argument, we have for constant $C_3 > 0$

$$|b_i^k| \le \begin{cases} C_3(1+|k+L|^\beta) & k \ge 0 \\ C_3(1+|k|^\beta) & k < 0 \end{cases} \tag{8}$$

Now for $M \in \mathbb{Z}$ and $M > 0$ define

$$f_M(x) = \sum_{k>M} (\boldsymbol{a}^k)^T \boldsymbol{\phi}(x-k) \qquad g_M(x) = \sum_{\ell>M} (\boldsymbol{b}^\ell)^T \boldsymbol{\phi}^*(x-\ell).$$

In an analogous manner define f_{-M} and g_{-M}. Then

$$\begin{aligned} f(x) &= \sum_{k=-M}^{M} (\boldsymbol{a}^k)^T \boldsymbol{\phi}(x-k) + f_{-M}(x) + f_M(x) \\ g(x) &= \sum_{\ell=-M}^{M} (\boldsymbol{b}^\ell)^T \boldsymbol{\phi}^*(x-\ell) + g_{-M}(x) + g_M(x) \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}} f(x)g(x)dx &= \int_{\mathbb{R}} \left(\sum_{k=-M}^{M} (\boldsymbol{a}^k)^T \boldsymbol{\phi}(x-k) \right) \left(\sum_{\ell=-M}^{M} (\boldsymbol{b}^\ell)^T \boldsymbol{\phi}^*(x-\ell) \right) dx \\ &\quad + \int_{\mathbb{R}} \left(\sum_{k=-M}^{M} (\boldsymbol{a}^k)^T \boldsymbol{\phi}(x-k) \right) (g_{-M}(x) + g_M(x))\, dx \\ &\quad + \int_{\mathbb{R}} \left(\sum_{\ell=-M}^{M} (\boldsymbol{b}^\ell)^T \boldsymbol{\phi}^*(x-\ell) \right) (f_{-M}(x) + f_M(x))\, dx \\ &\quad + \int_{\mathbb{R}} (f_{-M}(x) + f_M(x))(g_{-M}(x) + g_M(x))dx \\ &= \sum_{k=-M}^{M} (\boldsymbol{a}^k)^T \boldsymbol{b} + \int_{\mathbb{R}} (f_{-M}(x) + f_M(x))(g_{-M}(x) + g_M(x))dx \end{aligned}$$

Using (7), (8) and the fact that $\alpha - \beta > 1$ we see that the second term in the above sum tends to 0 as $M \to \infty$ so that we obtain the desired result. $\square$

We obtain the following moment formulas as immediate corollaries of Theorem 3.1.

Corollary 3.2 *Let $V(\boldsymbol{\phi})$ be generated by a compactly supported and stable $\boldsymbol{\phi}$ and let $\boldsymbol{\phi}^*$ be the dual of $\boldsymbol{\phi}$. Further assume that deg($V(\boldsymbol{\phi})) = n$,*

$$x^\beta = \sum_{k\in\mathbb{Z}} (\boldsymbol{c}^{k,\beta})^T \boldsymbol{\phi}^*(x-k), \qquad \beta = 0, \ldots, n, \tag{9}$$

and that $f \in V(\boldsymbol{\phi})$ satisfies the decay condition

$$|f(x)| \leq \frac{C}{1+|x|^{n+\alpha}}, \tag{10}$$

where $\alpha - \beta > 1$. Then

$$m_\beta(f) = \sum_{k\in\mathbb{Z}} (\boldsymbol{a}^k)^T \boldsymbol{c}^{k,\beta} \tag{11}$$

Corollary 3.2 illustrates how we may compute the moments of order β or less of $f \in V(\boldsymbol{\phi})$. Note that in order to implement (11), we must have the coefficient vector $\boldsymbol{c}^{k,\beta}$ for x^β. We will discuss a procedure for obtaining $\boldsymbol{c}^{k,\beta}$ later in the section.

The following corollary describes how we can refine our procedure and obtain moments $m_\beta(f)$ where $f \in V_h(\boldsymbol{\phi})$.

Corollary 3.3 *Let $f \in V_h(\boldsymbol{\phi})$ and have the representation*

$$f(x) = \sum_{k\in\mathbb{Z}} (\boldsymbol{a}^k)^T \boldsymbol{\phi}(\frac{x}{h} - k),$$

with $h > 0$. Further assume that deg($V_h(\boldsymbol{\phi})) = n$ and that f satisfies the growth condition (10) from Corollary 3.2. Let x^β be as in (9). Then

$$m_\beta(f) = h^{\beta+1} \sum_{k\in\mathbb{Z}} (\boldsymbol{a}^k)^T \boldsymbol{c}^{k,\beta} \tag{12}$$

We continue by deriving explicit representations for $\boldsymbol{c}^{k,\beta}$. Assume that $\deg(V(\boldsymbol{\phi})) = n$. Then for $\beta = 0, \ldots, n$, $j = 1, \ldots, r$, and $k \in \mathbb{Z}$ we have:

$$\begin{aligned} c_j^{k,\beta} &= \int_{\mathbb{R}} \phi_j(x-k) x^\beta dx \\ &= \int_{\mathbb{R}} \phi_j(x)(x+k)^\beta dx \\ &= \sum_{\ell=0}^{\beta} \begin{pmatrix} \beta \\ \ell \end{pmatrix} k^{\beta-\ell} \int_{\mathbb{R}} x^\beta \phi_j(x) dx \\ &= \sum_{\ell=0}^{\beta} \begin{pmatrix} \beta \\ \ell \end{pmatrix} k^{\beta-\ell} m_\beta(\phi_j) \end{aligned} \tag{13}$$

Thus to compute the $\boldsymbol{c}^{k,\beta}$, we need only the moments of order less than β of the components of $\boldsymbol{\phi}$. We shall see in the final section of the paper that in the case where $\boldsymbol{\phi}$ is refinable then this task can be performed recursively. Once we have these moments, we can use Corollary 3.2 or Corollary 3.3 to compute moments of functions in (finitely generated) principal shift invariant subspaces of $L^2(\mathbb{R})$. In addition Proposition 2.1 provides a means to estimate moments from these spaces should we intend to use them to approximate moments of functions in $L^2(\mathbb{R})$.

We conclude this section by noting that in light of (9) and (13) we have the means for establishing a Marsden's identity in any finitely generated shift invariant space of degree n. Of course for computational purposes, we must also obtain explicit formula for the moments $m_\beta(\phi_j)$, $\beta = 0, \ldots, n$ and $j = 1, \ldots, r$. Proposition 4.1 illustrates how we can obtain these values in the case where $r = 1$ and the function ϕ solves (14). We give examples of particular vector functions in the next section.

4 Refinable Functions and Vectors

In the final section of the paper, we discuss various methods for computing the initial moments $m_\beta(\phi_j)$ as given in (13). One of the most popular ways to obtain classes of (finitely generated) principal shift invariant spaces is to use ideas from wavelet or multiwavelet theory (see [2, 6] for wavelets, [8, 9] for multiwavelets). The idea is to construct a nested ladder of principal shift invariant subspaces of $L^2(\mathbb{R})$. This ladder is constructed by finding a function ϕ (or a vector $\boldsymbol{\phi}$) who along with its integer translates forms a Reisz basis for a space V_0. For $k \in \mathbb{Z}$, the space V_k is formed using the translates $\phi(2^k x - n)$, $n \in \mathbb{Z}$, of $\phi(2^k x)$. Other requirements are made of the nested ladder to ensure existence of a wavelet. The property we are particularly interested in is the refinement property (1).

Our first result of this section shows that if the generator ϕ is refinable, then we need only compute $\int \phi(x)dx$ and then use this value to recursively generate all moments needed in (13).

Proposition 4.1 *Assume that $\beta \geq 1$, $\beta \in \mathbb{Z}$, and that $\int_{\mathbb{R}} \phi(x)dx \neq 0$. Furthermore, suppose that there exists real numbers $p_0, \ldots, p_N$ so that*

$$\phi(x) = \frac{1}{2}\sum_{k=0}^{N} p_k \phi(2x-k). \tag{14}$$

Then

$$m_\beta(\phi) = \frac{1}{2(2^\beta - 1)} \sum_{\ell=0}^{\beta-1} \binom{\beta}{\ell} \left[\sum_{k=0}^{N} p_k k^{\beta-\ell}\right] m_\ell(\phi).$$

Proof. Multiply both sides of (14) by x^β and integrate over $\mathbb{R}$. Upon simplification, we obtain

$$m_\beta(\phi) = \frac{1}{(2^{\beta+1} - \sum_{k=0}^{N} p_k)} \sum_{\ell=0}^{\beta} \binom{\beta}{\ell} \left[\sum_{k=0}^{N} p_k k^{\beta-\ell}\right] m_\ell(\phi).$$

Take Fourier transforms of both sides of (14) and evaluate at 0. Since $\int_{\mathbb{R}} \phi(x)dx \neq 0$, we have

$$\frac{1}{2}\sum_{k=0}^{N} p_k = 1$$

from which the result follows. □

It is possible to generalize this result to the vector case. To this end, we introduce some new notation. Let $\boldsymbol{m}_\beta(\boldsymbol{\phi}) \in \mathbb{R}^{r\times r}$ be the vector whose components are given by $\boldsymbol{m}_\beta(\boldsymbol{\phi})_\ell = m_\beta(\phi_\ell)$, $\ell = 1, \ldots, r$. In addition we define the $r \times r$ matrix P by

$$P = \frac{1}{2}\sum_{k=0}^{N} P_k,$$

where the P_k satisfy the refinement condition (1).

Proposition 4.2 *Assume $\beta \geq 1$, $\beta \in \mathbb{Z}$ and that $\boldsymbol{\phi}$ satifies (1). Then $\boldsymbol{m}_\beta(\boldsymbol{\phi})$ can be obtained via recursion with $\boldsymbol{m}_0(\boldsymbol{\phi})$ as an initial starting point.*

Proof. Multiply both sides of (1) by x^β and integrate over $\mathbb{R}$. Upon simplification we have:

$$\boldsymbol{m}_\beta(\boldsymbol{\phi}) = 2^{-\beta-1} \sum_{j=0}^{\beta} \begin{pmatrix} \beta \\ j \end{pmatrix} \left[\sum_{k=0}^{N} P_k k^{\beta-j} \right] \boldsymbol{m}_j(\boldsymbol{\phi}).$$

Further simplification yields

$$(I - 2^{-\beta} P)\boldsymbol{m}_\beta(\boldsymbol{\phi}) = \frac{1}{2} \sum_{j=0}^{\beta-1} \begin{pmatrix} \beta \\ j \end{pmatrix} \left[\sum_{k=0}^{N} P_k k^{\beta-j} \right] \boldsymbol{m}_j(\boldsymbol{\phi}).$$

It is shown in [13] that σ_P, the spectral radius of P satisfies $\sigma_P = 1$. Thus $(I - 2^{-\beta}P)^{-1}$ exists. □

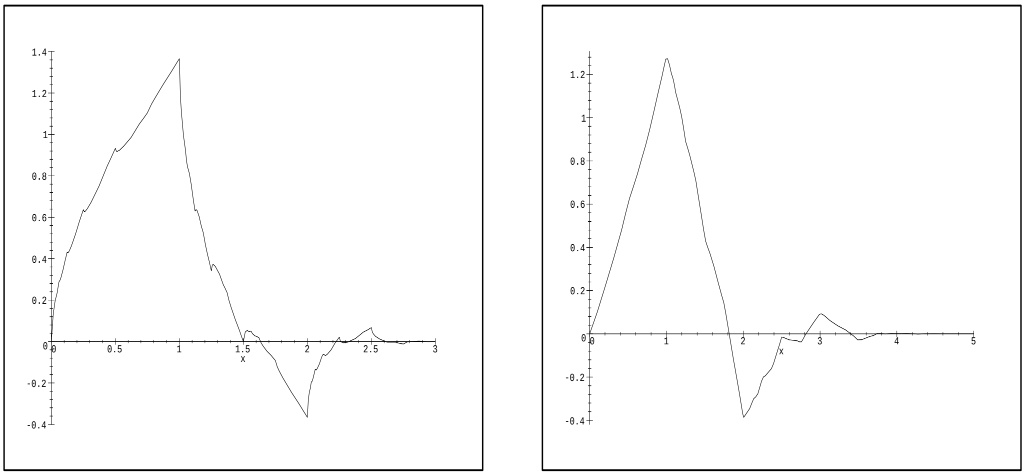
The propositions above illustrate that we can use functions from wavelet theory and multiwavelet theory to approximate moments of functions in $L^2(\mathbb{R})$. Daubechies ([6]) has created a family of functions that can possess arbitrary regularity. Chui and Wang ([4]) have derived wavelets from cardinal B-splines. In terms of multiwavelets, one could use Proposition 4.2 with the spline multiwavelets of Goodman and Lee ([9]) or the fractal multiwavelets given in ([7]).

We conclude the paper with two examples from the scaling functions listed in the previous paragraph. In both cases, we shall attempt to estimate moments of the *Dirichlet density*

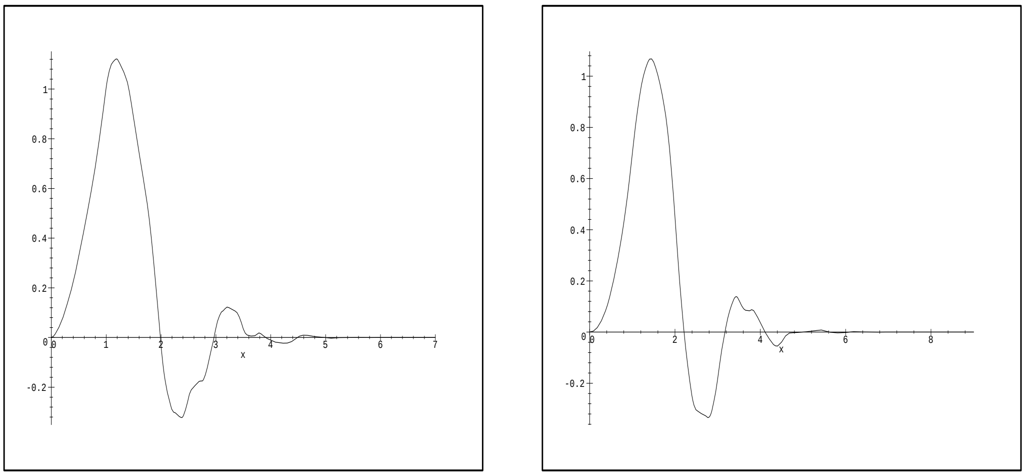
$$f(x; a, b) = \begin{cases} x^a(1-x)^b/B(a,b), & x \in [0,1] \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where $B(a, b)$ is the beta function and $a, b \in \mathbb{R}$ with $a, b > 0$.

Example 4.3 *We consider the family of Daubechies scaling functions $\phi^2, \ldots, \phi^5$ [6] (see Figure 1 below). V_0^ℓ is the closed linear span of the integer translates of ϕ^ℓ, $\ell = 2, \ldots, 5$. We take as V_j^ℓ the closed linear span of the set $\{2^{-j/2}\phi^\ell(2^j x - k)\}_{k \in \mathbb{Z}}$, $\ell = 2, \ldots, 5$. Note that $V_j^\ell \subset V_{j+1}^\ell$ and that deg$(V_j^\ell) = \ell - 2$.*



The functions ϕ^2 (left) and ϕ^3.



The functions ϕ^4 (left) and ϕ^5.

Figure 1. Scaling functions for Example 4.3.

We will estimate the $\beta = 1, 2, 3, 4$ moments of $f(\cdot; \frac{1}{2}, \frac{1}{2})$ using each of $\phi_2, \ldots, \phi_5$. In order to do so, we must calculate the 0 order moment for each scaling function. We can then use the recursion formula in Proposition 4.1 to compute the higher order moments that are used to form the $c^{k,\beta}$. The next step is to obtain the a^k's in Corollary 3.3. We provide our results for $h = 2^{-j}$, where $j = 6, 8, 10$.

We have provided three different methods for obtaining the a^k. In the first case, we simply sample f. In the second case, we approximate f by a piecewise quadratic polynomial and then use the precomputed $c^{k,\beta}$ to form a Newton-Cotes type integration scheme. The third method for computing the a^k uses the numerical integration from the prior method but uses a more sensitive approximation to f at the breakpoints $x = 0$ and $x = 1$.

Our results are given in the tables below. The numbers in parentheses represent the error between the approximation and the exact value. Note that we have used ϕ^2, ϕ^3, and ϕ^4 even in cases where Corollary 3.2 does not apply (that is, the order of the moment is larger than the degree of the space). The error in these cases is larger since x^β is not a member of the spaces generated by the corresponding scaling functions. In addition, since we must approximate the $\{a^k\}$ in some fashion the errors are dependent on the function f. Since f is only C^0, it it natural that the C^0 function ϕ^2 does an adequate job approximating the moments. Since the support of ϕ^2 is less than that of any other scaling function we use, the expansions for f_h consist of fewer terms. Thus, the computational cost of using ϕ^2 is less than that incurred by the other scaling functions.

The actual moments of f for this example are $m_1(f) = .5$, $m_2(f) = .3125$, $m_3(f) = .21875$, and $m_4(f) = .1640625$.

Daubechies' ϕ^2 function.

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.50230275(+4.61e-03)	0.50055824(+1.12e-03)	0.50013547(+2.71e-04)
2	0.30729056(-1.67e-02)	0.31112133(-4.41e-03)	0.31214812(-1.13e-03)
3	0.21142351(-3.35e-02)	0.21684828(-8.69e-03)	0.21826775(-2.20e-03)
4	0.15614446(-4.83e-02)	0.16200804(-1.25e-02)	0.16354169(-3.17e-03)

Daubechies' ϕ^3 function.

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.48950970(-2.10e-02)	0.49736740(-5.27e-03)	0.49933799(-1.32e-03)
2	0.29483178(-5.65e-02)	0.30795063(-1.46e-02)	0.31135186(-3.67e-03)
3	0.20000283(-8.57e-02)	0.21389255(-2.22e-02)	0.21752234(-5.61e-03)
4	0.14570952(-1.12e-01)	0.15926406(-2.92e-02)	0.16284690(-7.41e-03)

Daubechies' ϕ^4 function.

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.47679350(-4.64e-02)	0.49419456(-1.16e-02)	0.49854499(-2.91e-03)
2	0.28276572(-9.51e-02)	0.30481785(-2.46e-02)	0.31056136(-6.20e-03)
3	0.18911003(-1.35e-01)	0.21098329(-3.55e-02)	0.21678298(-8.99e-03)
4	0.13589634(-1.72e-01)	0.15657257(-4.57e-02)	0.16215836(-1.16e-02)

Daubechies' ϕ^5 function.

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.46408879(-7.18e-02)	0.49102384(-1.80e-02)	0.49775250(-4.49e-03)
2	0.27102926(-1.33e-01)	0.30170717(-3.45e-02)	0.30977261(-8.73e-03)
3	0.17867438(-1.83e-01)	0.20810555(-4.87e-02)	0.21604598(-1.24e-02)
4	0.12662787(-2.28e-01)	0.15391947(-6.18e-02)	0.16147259(-1.58e-02)

Table 1: a^k sampled from f.

Daubechies' ϕ^2 function.

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.51540693(+3.08e-02)	0.50388603(+7.77e-03)	0.50097426 (+1.95e-03)
2	0.32038575(+2.52e-02)	0.31444970(+6.24e-03)	0.31298696(+1.56e-03)
3	0.22358158(+2.21e-02)	0.21995940(+5.53e-03)	0.21905338(+1.39e-03)
4	0.16737420(+2.02e-02)	0.16490249(+5.12e-03)	0.16427415(+1.29e-03)

Daubechies' ϕ^3 function.

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.51796103(+3.59e-02)	0.50386543(+7.73e-03)	0.50097173(+1.94e-03)
2	0.32299014(+3.36e-02)	0.31442920(+6.17e-03)	0.31298446(+1.55e-03)
3	0.22602114(+3.32e-02)	0.21993897(+5.44e-03)	0.21905087(+1.38e-03)
4	0.16964373(+3.40e-02)	0.16488212(+5.00e-03)	0.16427162(+1.27e-03)

Daubechies' ϕ^4 function.

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.51797521(+3.60e-02)	0.50385567(+7.71e-03)	0.50097056(+1.94e-03)
2	0.32301051(+3.36e-02)	0.31441960(+6.14e-03)	0.31298329(+1.55e-03)
3	0.22606401(+3.34e-02)	0.21992972(+5.39e-03)	0.21904971(+1.37e-03)
4	0.16970560(+3.44e-02)	0.16487322(+4.94e-03)	0.16427047(+1.27e-03)

Daubechies' ϕ^5 function.

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.51794667(+3.59e-02)	0.50384989(+7.70e-03)	0.50096988(+1.94e-03)
2	0.32298667(+3.36e-02)	0.31441390(+6.12e-03)	0.31298263(+1.54e-03)
3	0.22608993(+3.36e-02)	0.21992478(+5.37e-03)	0.21904904(+1.37e-03)
4	0.16977257(+3.48e-02)	0.16486901(+4.92e-03)	0.16426981(+1.26e-03)

Table 2: a^k obtained by numerical integration.

Daubechies' ϕ^2 function.

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.50992095(+1.98e-02)	0.50388040(+7.76e-03)	0.50097352(+1.95e-03)
2	0.31482643(+7.44e-03)	0.31444359(+6.22e-03)	0.31298622(+1.56e-03)
3	0.21857369(-8.06e-04)	0.21995342(+5.50e-03)	0.21905261(+1.29e-03)
4	0.16291963(-6.97e-03)	0.16489665(+5.08e-03)	0.16427338(+1.29e-03)

Daubechies' ϕ^3 function.

β	$h=2^{-6}$	$h=2^{-8}$	$h=2^{-10}$
1	0.51390530(+2.78e-02)	0.50356705(+7.13e-03)	0.50093456(+1.87e-03)
2	0.31910721(+2.13e-02)	0.31413720(+5.24e-03)	0.31294749(+1.43e-03)
3	0.22229573(+1.62e-02)	0.21964928(+4.11e-03)	0.21901395(+1.21e-03)
4	0.16602164(+1.19e-02)	0.16459481(+3.24e-03)	0.16423478(+1.05e-03)

Daubechies' ϕ^4 function.

β	$h=2^{-6}$	$h=2^{-8}$	$h=2^{-10}$
1	0.50995077(+1.99e-02)	0.50312468(+6.25e-03)	0.50087968(+1.76e-03)
2	0.31583967(+1.07e-02)	0.31371336(+3.88e-03)	0.31289319(+1.26e-03)
3	0.21913903(+1.78e-03)	0.21923114(+2.20e-03)	0.21895985(+9.59e-04)
4	0.16305911(-6.12e-03)	0.16418253(+7.32e-04)	0.16418086(+7.21e-04)

Daubechies' ϕ^5 function.

β	$h=2^{-6}$	$h=2^{-8}$	$h=2^{-10}$
1	0.50516894(+1.03e-02)	0.50257991(+5.16e-03)	0.50081227(+1.62e-03)
2	0.31214977(-1.12e-03)	0.31320175(+2.25e-03)	0.31282678(+1.05e-03)
3	0.21570730(-1.39e-02)	0.21872938(-9.43e-05)	0.21889378(+6.57e-04)
4	0.15993477(-2.52e-02)	0.16369097(-2.26e-03)	0.16411512(+3.21e-04)

Table 3: a^k obtained by adaptive numerical integration.

We now consider an example illustrating our methods with finitely generated shift invariant spaces. To this end, we employ the scaling vector comprised of fractal interpolation functions given in ([7]). We also note that in the case of these functions, a recursion formula for the moments exists (see [14]) and could be used in place of Proposition 4.2.

Example 4.4 *We consider the closed linear space V_0 spanned by the fractal interpolation functions ϕ_1 and ϕ_2 as derived in ([7]). We choose ϕ_1, ϕ_2 so that $deg(V_0) = 3$. We obtain the V_j spaces by taking the closed linear span of the set $\{2^{-j/2}\phi_\ell(2^j x - k), \ell = 1, 2\}_{k\in\mathbb{Z}}$. As in Example 4.3, we approximate moments of the beta distribution.*

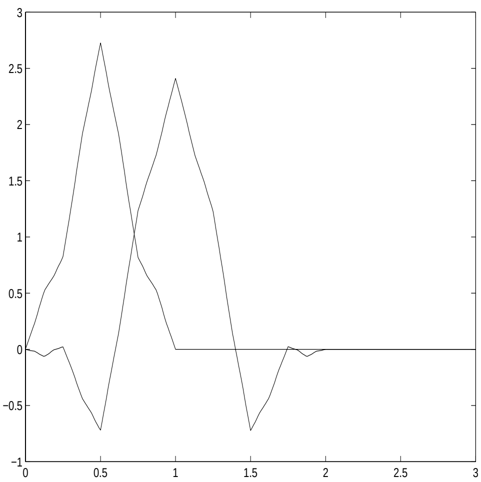


Figure 2. The fractal interpolation functions ϕ_1 and ϕ_2.

Note that the accuracy is about the same as that of the Daubechies ϕ^2 function. In the first table, the a_k were function samples; in the second table the a_k were obtained using numerical integration; in the third table the a_k were obtained using adaptive numerical integration. The adaptive integration is not as effective here since one of the scaling functions has the same support as does f. The actual values for $m_\beta(f)$, $\beta = 1, 2, 3, 4$, are given in the previous example.

a_k from function samples:

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
hline 1	0.51082092(+2.16e-02)	0.50306346(+6.13e-03)	0.50107940(+2.16e-03)
2	0.31564133(+1.01e-02)	0.31347012(+3.10e-03)	0.31293711(+1.40e-03)
3	0.21909522(+1.58e-03)	0.21898027(+1.05e-03)	0.21894533(+8.93e-04)
4	0.16317880(-5.39e-03)	0.16395625(-6.48e-04)	0.16414017(+4.73e-04)

a_k obtained via numerical integration:

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.51494589(+2.99e-02)	0.50382728(+7.65e-03)	0.50096687(+1.93e-03)
2	0.31996907(+2.39e-02)	0.31439355(+6.06e-03)	0.31297975(+1.54e-03)
3	0.22319232(+2.03e-02)	0.21990482(+5.28e-03)	0.21904625(+1.35e-03)
4	0.16700720(+1.79e-02)	0.16484914(+4.79e-03)	0.16426708(+1.25e-03)

a_k obtained via adaptive numerical integration:

β	$h = 2^{-6}$	$h = 2^{-8}$	$h = 2^{-10}$
1	0.51444314(+2.89e-02)	0.50376013(+7.52e-03)	0.50095837(+1.92e-03)
2	0.31946792(+2.23e-02)	0.31432651(+5.84e-03)	0.31297124(+1.51e-03)
3	0.22269420(+1.80e-02)	0.21983786(+4.97e-03)	0.21903774(+1.32e-03)
4	0.16651210(+1.49e-02)	0.16478230(+4.39e-03)	0.16425857(+1.20e-03)

Table 2. Moment computation from Example 4.4.

References

[1] C. deBoor, R. DeVore, and A. Ron, "Approximation from shift- invariant subspaces of $L_2(\mathbb{R}^d)$, Trans. Amer. Math. Soc., **341** (1994), 787–806.

[2] C.K. Chui, *An Introduction to Wavelets*, Academic Press, San Diego, 1992.

[3] C.K. Chui and M.J. Lai, "A multivariate analog of Marsden's identity and a quasi-interpolation scheme", Constr. Approx., **3** (1987), 111–122.

[4] C.K. Chiu and J.Z. Wang, "A general framework of compactly supported splines and wavelets", J. Approx. Th., **71**(1992), 263-304.

[5] A. Cohen, I. Daubechies, G. Plonka, "Regularity of refinable function vectors", (preprint).

[6] I. Daubechies, *Ten Lectures on Wavelets*, SIAM CBMS Series No. 61, Philadelphia, 1992.

[7] G. Donovan, J. Geronimo, D. Hardin, and P. Massopust, Construction of orthogonal wavelets using fractal interpolation functions", (to appear SIAM J. Math. Anal.).

[8] J. Geronimo, D. Hardin, and P. Massopust, "Fractal functions and wavelet expansions based on several scaling functions", J. Approx. Theory, **78**(3) (1994), 373-401.

[9] T.N.T Goodman, and S. L. Lee, "Wavelets of multiplicity r", Trans. Amer. Math. Soc., Vol. 342, Number 1, March 1994, 307-324.

[10] C. Heil, G. Strang, V. Strela, "Approximation by translates of refinable functions, Numer. Math., (to appear).

[11] Jia, R.Q., "Shift-Invariant spaces on the real line", Proc. Amer. Math. Soc., (to appear).

[12] M.J. Marsden, "An identity for spline functions with applications to variation-diminishing spline approximation", J. Approx. Th., **3** (1970), 7–49.

[13] P.R. Massopust, D.K. Ruch, and P.J. Van Fleet, "On the support properties of scaling vectors", Comp. and Appl. Harm. Anal., **3**(1996), 229–238.

[14] P.R. Massopust and P.J. Van Fleet, "On the moments of fractal functions and Dirichlet spline functions", (preprint).

[15] G. Plonka, "Approximation properties of multi-scaling functions: A Fourier approach", Constr. Approx. (to appear).