On the Evaluation of Simplicial Splines

Patrick J. Van Fleet*

Abstract

We consider the problem of large-scale evaluation of multivariate simplicial splines. These splines arise naturally as the multivariate analog of B-splines and are useful in applications such as finite element methods and computer-aided geometric design (CAGD). Simplicial splines have much in common with their univariate analog. They are piecewise polynomial functions whose smoothness and support can be controlled by knot placement. In addition, these splines obey a recurrence formula. Unlike the univariate case, implementation of this formula poses some problems. If $x \in \mathbb{R}^s$ lies on an $(s-1)$-dimensional hyperplane $\mathcal{H}$ connecting two or more knot points, then the recurrence formula cannot be utilized.

Results on the evaluation of multivariate simplicial splines are documented in [7, 10]. In both cases, the authors use perturbation techniques when faced with evaluating the spline at $x \in \mathcal{H}$. In [7], the author suggests a stable means for evaluating simplicial splines at points on the interior of regions formed by $(s-1)$-dimensional hyperplanes and we augment this method with a scheme for evaluating points on any such hyperplane. Our method is especially suited to large-scale evaluation of the spline and it is numerically stable when the evaluation is performed near or on $(s-1)$-dimensional hyperplanes. Examples are given to demonstrate the algorithm.

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*Department of Mathematics, Sam Houston State University, Huntsville, TX 77341 USA, vanfleet@guy.shsu.edu.

1 Introduction

We consider the problem of evaluation multivariate simplicial splines. These splines are the natural multivariate extension of the B-spline of Curry and Schoenburg [5]. These splines can be defined using a geometric interpretation or via a distributional rule that arises when realizing the spline is the multiplier in an inner product that defines a linear functional. Neither definition allows for easy evaluation. Micchelli [8] proved that a recurrence formula exists for these splines and that a simplicial spline of total degree n can be expressed as a linear combination of simplicial splines of degree $n-1$. This formula can be repeated until the spline is ultimately expressed in terms of constant functions.

This recurrence has been studied by Grandine [7] and Neamtu [10]. Both authors realized that this formula has problems. The linear combinations at each level require knowledge of the barycentric coordinates of the evaluation point. Moreover if the evaluation point lies on an $(s-1)$-dimensional hyperplane connecting two or more knot points, the recursion formula of deBoor is invalid.

Grandine [7] handled the problem of determining barycentric coordinates by using linear programming methods. He was also able to avoid the hyperplane problem in the case where the knot points of the spline were in general position. In this paper, we generalize the work of Grandine (in the bivariate case) and provide a method for evaluating a simplicial spline at a large number of points. The algorithm will work for any configuration of knot points. Hyperplane problems are avoided by appealing to the regularity properties possessed by the splines and utilizing a bivariate interpolation scheme due to Micchelli [9].

The outline of this paper is as follows: In Section 2, we give basic definitions, notation, and elementary results necessary to the sequel. The derivation of multivariate simplicial splines is given in the next section as well as the recursion formula of Micchelli [8]. Grandine's method is outlined in Section 4. The final section contains the generalization of this algorithm as well as several examples illustrating our results.

2 Notation, Definitions, and Basic Results

In this section we introduce notation, definitions, and basic results used throughout the remainder of the paper.

Let A be a nonempty set in $\mathbb{R}^s$. Then we make the following definitions. $\chi_A(x)$ is the characteristic function on A, $[A]$ is its convex hull of A, and $\text{vol}_k(A)$ is its k-dimensional volume of A.

Other sets of interest are $\pi_n(\mathbb{R}^2)$ – the set of all bivariate polynomials of degree less than or equal to n and the Euclidean n-simplex S^n given by:

$$S^n = \{(t_1, \ldots, t_n) : \sum_{k=1}^{n} t_k \leq 1, t_k \geq 0\}.$$

We define the residual $t_0 = 1 - t_1 - t_2 - \cdots - t_n$.

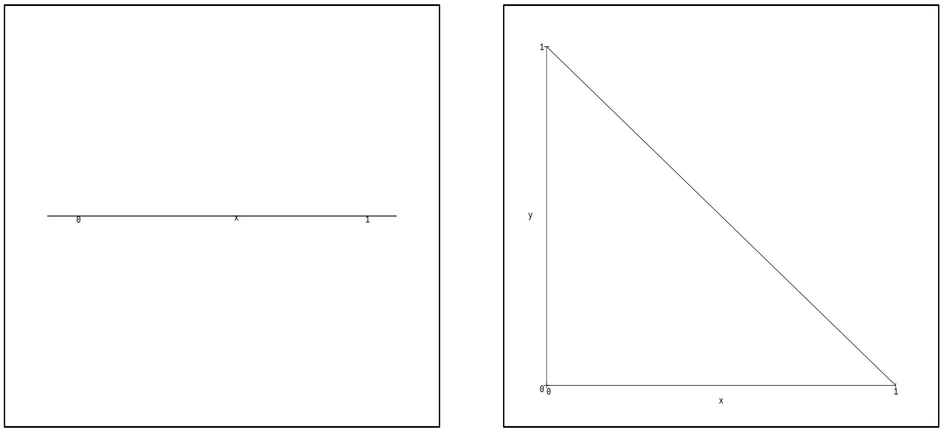

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Figure 1: The simplices S^1 and S^2.

In order to define univariate B-splines we will need the divided difference operator $[t_0, \ldots, t_n]g$, of order n acting on function g (see for example [1]), and the truncated power function $y_+ = \max\{0, y\}$.

For multivariate splines, we will consider knot sets $X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^s$, $n \geq s$, and make use of the set $X_k = X \backslash \{x^k\}$.

3 Simplicial Splines Defined

In this section, we restate Micchelli's derivation of the definition of multivariate simplicial splines as well as list several properties obeyed by these splines. Since univariate B-splines are defined in terms of divided differences, the generalization to the higher dimensional setting is difficult. Neamtu [10] has given a nice definition of multivariate divided differences, but the definition does not allow all the properties of univariate B-splines to carry over to the multivariate setting. deBoor's observation that the univariate B-splines measure n dimensional volume of a certain simplex provides the best bridge to higher dimensions.

Let $t_0 \leq \cdots \leq t_n$, with $t_0 < t_n$. We define the univariate *B-spline of order n* by:

$$N_{0,n}(x) = n[t_0, \ldots, t_n](t - x)_+^{n-1}, \tag{1}$$

where the divided difference operator acts on t. Some basic properties of B-splines have been established (see deBoor [2]):

1. $\int_{\mathbb{R}} N_{0,n}(x)dx = 1$.

2. $N_{0,n}(x) \geq 0$, for all x.

3. $\text{supp}(N_{0,n}(x)) = [t_0, t_n]$.

4. $N_{0,n}$ is a piecewise polynomial of degree $\leq n$ with possible breakpoints at the knots $t_0, \ldots, t_n$.

5. If t is not a knot point, then $N_{0,n}$ is $n-2$ times continuously differentiable. If t_i is repeated k times, then $N_{0,n}$ is $n-k-1$ times continuously differentiable.

The B-spline $N_{0,n}$ also satisfies a recursion formula due to deBoor [1]:

Theorem 3.1 *Let $N_{0,n}$ be given as in (1). Then*

$$\frac{n-1}{n} N_{0,n}(x) = \frac{x-t_0}{t_n-t_0} N_{0,n-1}(x) + \frac{t_n-x}{t_n-t_0} N_{1,n-1}(x), \tag{2}$$

at all points x where the splines in (2) are defined and continuous.

Using the Hermite-Gennocchi formula for divided differences, we can show that $N_{0,n}$ obeys the following distributional relationship:

$$\int_{\mathbb{R}} N_{0,n}(x) g(x) dx = n! \int_{S^n} g(\lambda_0 t_0 + \cdots \lambda_n t_n) d\lambda, \tag{3}$$

where $\lambda_0 = 1 - \lambda_1 - \cdots - \lambda_n$ and g is any continuous function on $\mathbb{R}$ with compact support.

The following derivation is due to Micchelli [8]:

Let $Y = \{y^0, \ldots, y^n\}$ be a set of vectors in $\mathbb{R}^n$ such that $\text{vol}_n[Y] = 1$ and the first component of each y^k satisfies

$$y_1^k = t_k, \qquad k = 0, \ldots, n.$$

Set $u = \lambda_0 y^0 + \cdots \lambda_n y^n$, where $\lambda = (\lambda_1, \ldots, \lambda_n)^T \in S^n$. Note then that the Jacobian of this transformation is 1 so that (3) becomes

$$\begin{aligned} \int_{\mathbb{R}} N_{0,n}(x) g(x) dx &= n! \int_{\mathbb{R}^n} \chi_{([Y])}(u_1, \ldots, u_n) du \\ &= \int_{\mathbb{R}} \left(n! \int_{\mathbb{R}^{n-1}} \chi_{[Y]}(x, u_2, \ldots, u_n) du_2 \cdots du_n \right) g(x) dx. \end{aligned}$$

Thus we see that

$$N_{0,n}(x) = n! \text{vol}_{n-1}\{y \in [Y] : y_1 = x\}. \tag{4}$$

The following example illustrates this point:

Example 3.2 *Let $n = 2$ with $t_0 = 0$, $t_1 = \frac{1}{2}$, and $t_2 = 1$. We choose as our simplex $y^0 = (0,0)^T$, $y^1 = (\frac{1}{2},2)^T$, and $y^2 = (1,0)^T$. Then $N_{0,2}(x)$ is simply the length of line segment L.*

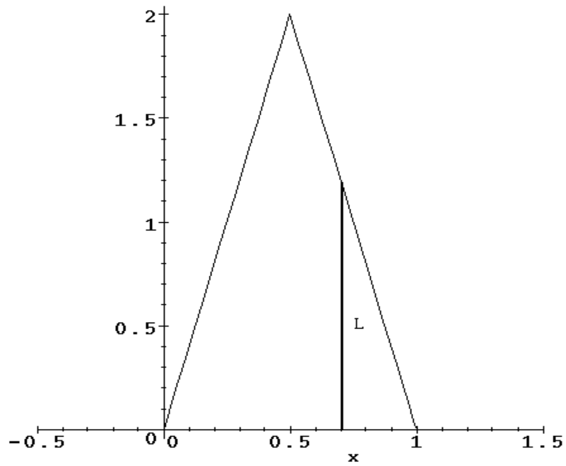

Figure 2. The linear B spline from example 3.2

Note that the identity (4) provides a natural way to define multivariate simplicial splines. Let $X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^s$, $n \geq s$ with $\text{vol}_s[X] > 0$ and assume $Y = \{y^0, \ldots, y^n\} \subset \mathbb{R}^n$ with $\text{vol}_s[Y] = 1$. Then we define the *multivarate simplicial spline* $M(x|X)$ as

$$M(x|X) = n!\text{vol}_{n-s}\{y \in [Y] : y_j = x_j, j = 1, \ldots, s\}. \tag{5}$$

The definition allows for all of the properties of univariate B-splines to hold in the higher dimensional setting (see Micchelli [8]):

1. If $n = s$ then
$$M(x|X) = \frac{1}{\text{vol}_s[X]} \chi_{\text{int}([X])}(x). \tag{6}$$

2. $\int_{\mathbb{R}^s} M(x|X)dx = 1$.

3. $M(x|X) \geq 0$.

4. $\text{supp}(M) = [X]$.

5. $M(\cdot|X)$ is a piecewise polynomial of total degree $\leq n - s$.

6. Let $k \geq s$ be the maximal number of knots that lie in the same $(s-1)$-dimensional hyperplane. Then
$$M(\cdot|X) \in C^{n-k-1}(\mathbb{R}^s). \tag{7}$$

The multivariate simplicial spline also satisfies a distributional relation:

$$\int_{\mathbb{R}^s} f(x)M(x|X)dx = n! \int_{S^n} f(\lambda_0 x^0 + \cdots + \lambda_n x^n)d\lambda \tag{8}$$

for all $f \in C(\mathbb{R}^s)$ with compact support.

Micchelli [8] showed that multivariate simplicial splines also obey a recursion formula:

Theorem 3.3 *Suppose $x = \sum_{k=0}^n \lambda_k x^k$, $\sum_{k=0}^n \lambda_k = 1$, and $\lambda_k \geq 0$, $k = 0, \ldots, n$. Then*

$$M(x|X) = \frac{n}{n-s} \sum_{k=0}^{n} \lambda_k M(x|X_k) \tag{9}$$

at all points x where the splines on the right hand side of (3.3) are continuous.

We conclude this section with an example illustrating the properties of multivariate simplicial splines.

Example 3.4 *Let $n = 3$ and $X = \{(0,0)^T, (0,0)^T, (1,0)^T, (0,1)^T\}$. The degree of the spline is 1 and $k = 3$ from 7 above. So M is bounded only.*

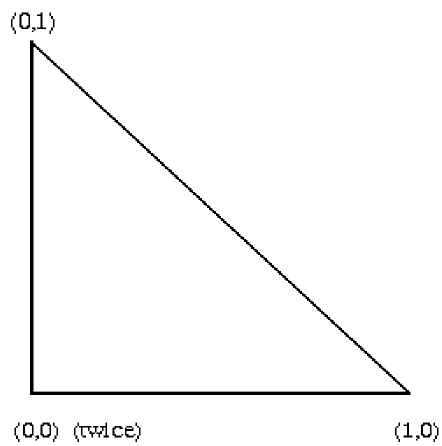

Figure 3. *The domain of the spline from Example 3.4*

Example 3.5 *Let $n = 5$ and $X = \{(0,0)^T, (0,0)^T, (1,0)^T, (1,0)^T, (0,1)^T, (0,1)^T\}$. The degree of the spline is 3 and $k = 4$ from 7 above. So M is continuous.*

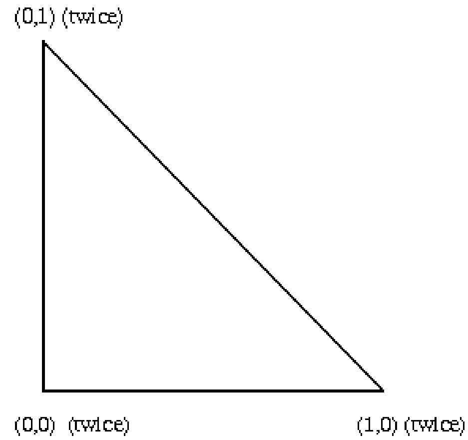

Figure 4. *The domain of the spline from Example 3.5*

Example 3.6 *Let* $n = 3$ *and* $X = \{(1,1)^T, (1,-1)^T, (-1,-1)^T, (-1,1)^T\}$. *The degree of the spline is* 1 *and* $k = 2$ *from 7 above. So* M *is continuous.*

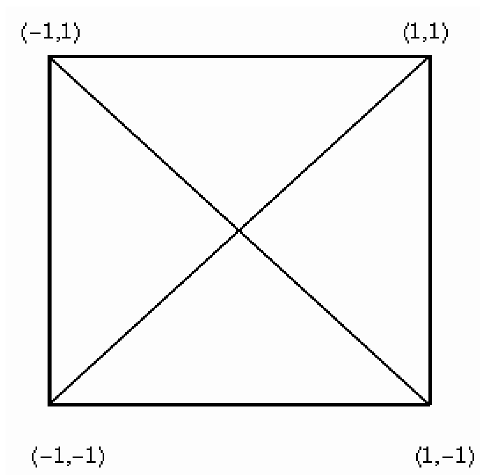

Figure 5. *The domain of the spline from Example 3.6*

Example 3.7 *Let* $n = 3$ *and* $X = \{(1,1)^T, (1,-1)^T, (-1,-1)^T, (-1,1)^T, (0,0)^T\}$. *Then the degree of the spline is* 2 *and* $k = 3$ *from 7 above. So* M *is continuous.*

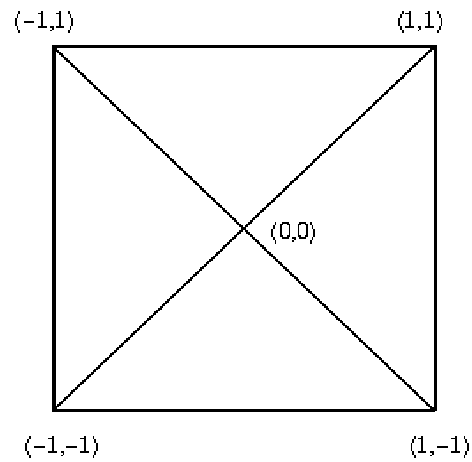

Figure 6. *The domain of the spline from Example 3.7*

4 Grandine's Method for Numerically Evaluating Simplicial Splines

In order to understand how Grandine's method works, recall the requirements for x in Theorem 3.3:

$$\begin{aligned} \sum_{k=0}^{n} \lambda_k x^k &= x \\ \sum_{k=0}^{n} \lambda_k &= 1 \\ \lambda_0, \ldots, \lambda_n &\geq 0. \end{aligned} \tag{10}$$

In order to implement the recursion formula (9), we must have the ability to compute the λ_k's for a given x. The system (10) can be viewed as the feasible region of a linear programming problem and finding the λ_k's is equivalent to pivoting to a point in the feasible region. If we cannot pivot into the feasible region, it follows immediately that $x \notin [X]$ and thus by Property 3 in the previous section $M(x|X) = 0$.

As far as producing a solution to (10), the two-phase method outlined in [6] works well. We illustrate with the following example.

Example 4.1 *Let X be the knot set given in Example 3.6 and suppose we wish to evaluate the resulting spline at the point $(\frac{1}{2}, 0)^T$.*

The recursion formula (9) becomes:

$$M(x|X) = 3(\lambda_0 M(x|X_0) + \lambda_1 M(x|X_1) + \lambda_2 M(x|X_2) + \lambda_3 M(x|X_3). \tag{11}$$

Using Property 1 of the multivariate splines we see that

$$M(x|X) = \begin{cases} 1 & x \in \operatorname{int}([X_k]) \\ 0 & \text{otherwise.} \end{cases}$$

for each $k = 0, 1, 2, 3$. Upon inspection of Figure 4, we see that $M(x|X_0) = M(x|X_1) = 0$ so that (11) reduces to

$$M(x|X) = 3(\lambda_2 + \lambda_3).$$

Thus, we need only determine λ_2 and λ_3 in order to evaluate the spline. We consider the following linear programming problem:

Minimize $C = a_1 + a_2 + a_3$
such that

$$\begin{aligned} \lambda_0 + \lambda_1 - \lambda_2 - \lambda_3 + a_1 &= \frac{1}{2} \\ \lambda_0 - \lambda_1 - \lambda_2 + \lambda_3 + a_2 &= 0 \\ \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + a_3 &= 1 \\ \lambda_0, \ldots, \lambda_3, a_1, a_2, a_3 &\geq 0 \end{aligned} \tag{12}$$

Here, a_1, a_2, a_3 are artificial variables introduced so that the minimum value of C is 0. In the process of forcing the a_k's to zero, the two-phase method returns $\lambda_0 = \frac{1}{2}$, $\lambda_1 = \lambda_2 = \frac{1}{4}$, and $\lambda_3 = 0$. Thus we have

$$M(x|X) = \frac{3}{4}.$$

There are two advantages to Grandine's scheme. First, the two-phase method is stable. The second advantage is that the two-phase method works for any spatial dimension. The disadvantages to this scheme are that it is extremely costly to perform large scale evaluation of the spline when the simplex method has to be called many times. A greater disadvantage is that the scheme fails if the evaluation point x lies on an $(s-1)$-dimensional hyperplane connecting s or more knot points. We illustrate this disadvantage with the following example:

Example 4.2 *Suppose we wish to evaluate the spline $M(\cdot|X)$ from Example 3.6 at the point $x = (0,0)^T$. Clearly we can choose $\lambda_0 = \lambda_2 = \frac{1}{2}$ and $\lambda_1 = \lambda_3 = 0$ so that*

$$M(x|X) = 3\{\frac{1}{2}M(x|X_0) + \frac{1}{2}M(x|X_2)\}.$$

We cannot use Property 1 in the previous section to evaluate the constant splines. In the univariate case, we avoid this problem by *a priori* deciding that the constant function will be left continuous or right continuous. This is not so in higher dimensions. Grandine [7] does suggest a method for solving this problem, but he requires that the original knot points be in general position. A set $X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^s$ is said to be in *general position* if the maximum number of knot points that lie on any $(s-1)$-dimensional hyperplane is s. General position eliminates repeated knots or *interior* knots that might lie on a hyperplane connecting s *exterior* knots. For completeness, we include a pseudocode of Grandine's algorithm. To this end, we introduce new notation. Let $N = \{0, \ldots, n+1\}$ and for $k_0, \ldots, k_j \in \mathbb{N}$, the symbol $X_{k_0,\ldots,k_j}$ denotes the set $X \backslash \{x^{k_0}, \ldots, x^{k_n}\}$. Here we assume $0 \leq j \leq n$, and $k_0, \ldots, k_j$ are distinct integers.

Algorithm 4.3 *Given $X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^s$, $n > s$, $vol_s[X] > 0$, and $x \in \mathbb{R}^s$, with x not contained in any $(s-1)$-dimensional hyperplane connecting s or more knot points, this algorithm produces the value $M(x|X)$. Use the two-phase method to determine all sets of barycentric coordinates.*

```
Compute barycentric coordinates λ_{k_0}, k_0 ∈ IN, for x relative to X
T_0 = 0
For k_0 do
    Compute the barycentric coordinates λ_{k_0,k_1}, k_1 ∈ IR\{k_0}, for x relative to X_{k_0}
    T_1 = 0
    For k_1 do
        Compute the barycentric coodinates λ_{k_0,k_1,k_2}, k_2 ∈ IN\{k_0,k_1}, for x relative to X_{k_0,k_1}
        T_2 = 0
        ⋮
            ⋮
            For k_{n-s-2} do
                Compute barycentric coordinates λ_{k_0,...,k_{n-s-1}}, k_{n-s-1} ∈ IN\{k_0,...,k_{n-s-2}},
                for x relative to X_{k_0,...,k_{n-s-2}}
                T_{n-s-1} = 0
                For k_{n-s-1} do
                    Use 6 to evaluate M(x|X_{k_0,...,k_{n-s-1}})
                    T_{n-s-1} = T_{n-s-1} + ((s+1)/s) M(x|X_{k_0,...,k_{n-s-1}})
                T_{n-s-2} = T_{n-s-2} + ((s+2)/2) T_{n-s-1}
            ⋮
        ⋮
    T_1 = T_1 + ((n-s-1)/(n-1)) T_2
    T_0 = T_0 + ((n-s)/n) T_1
    M(x|X) = T_0
```

5 A Bivariate Generalization of Grandine's Method

In this section we present a bivariate generalization of Grandine's method. While this scheme only requires that $\mathrm{vol}_2([X]) > 0$, it is unsuitable for evaluating the spline at a few points. Indeed, our procedure is intended for large scale evaluation of the simplicial spline.

The idea is to exploit Property 4 from Section 3. That is, for $X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^2$,

$M(\cdot|X)$ is a piecewise polynomial of total degree $\leq n-2$. Moreover, the domains of the polynomial pieces are the regions formed by the line segments connecting knot points.

Since a bivariate polynomial of degree $n-2$ consists of at most $\binom{n}{2}$ terms, we begin our algorithm by using Algorithm 4.3 to evaluate $\binom{n}{2}$ *interior* points of each region. Since we stay away from boundary points, Algorithm 4.3 works well. We then appeal to a bivariate interpolation theorem of Micchelli [9] to define the piecewise polynomial on each region. All that is left is to determine which region houses a given x and then evaluate the corresponding polynomial. If x resides on the boundary of a region, we apply the regularity Property 5 from Section 3 to determine how $M(x|X)$ should be evaluated.

Along with Grandine's method and Micchelli's interpolation theorem, we need some additional computational tools. First we need to know an upper bound on the number of regions that can be formed from $n+1$ points in the plane. Next, we need a way to characterize each of these regions and then use this information to determine which region holds x.

The following proposition gives some insight to the number of regions formed by $n+1$ points.

Proposition 5.1 *Let $X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^2$. Denote by $R(n)$ the number of regions formed by connecting all points in X. Then $R(0) = R(1) = 0$, $R(2) = 1$, and for $n > 2$,*

$$R(n) \leq R(n-2) + \binom{n}{3} + n. \tag{13}$$

Moreover if the points of X are in general position, equality holds in (13).

Proof. If the points of X are in general position, then the proof of the equality of (13) can be found in [4]. Otherwise at least one point lies on a line segment connecting two other points. Let the number of these points be denoted by k. Then it easy to see that $R(n) = R(n-k)$. It is easy to show by induction that R is a nondecreasing function so that the inequality in (13) holds. □

Now that we know the total number of regions formed by the points in X, we need a method for characterizing points in these regions. Each line segment can be expressed as $ax + by + c = 0$ so that a point $x = (x_1, x_2) \in \mathbb{R}^2$ must either satisfy

$$ax_1 + bx_2 + c \geq 0 \tag{14}$$

or

$$ax_1 + bx_2 + c < 0. \tag{15}$$

There are $\binom{n}{2}$ such lines. We form an $\binom{n}{2} \times \binom{n}{2}$ matrix A^i for each region i and define the elements A^i_{rs} by:

$$A^i_{rs} = \begin{cases} 1 & \text{if (14) holds for the line connecting } x^r, x^s, \\ -1 & \text{if (14) holds for the line connecting } x^r, x^s. \end{cases}$$

Note that each A^i is symmetrical and we need only compute A^i_{rs} for $r < s$.

Now we have an interpolation problem to solve on each region. The polynomial interpolant will be exact if we use a polynomial of degree $n-2$. Our original attempt to construct this interpolant was to form the Vandermonde matrix and solve the system. But for randomly chosen points in a region, this procedure is highly unstable. We avoided this problem by showing that the conditions of the following theorem can be satisfied.

Theorem 5.2 [Micchelli]. *Let $\ell_0, \ldots, \ell_m$ be lines such that each pair of lines intersect at a point and every point lies on exactly two lines. If x^j, $j = 1, \ldots, N$, $N = \binom{m}{2}$, are the intersection points, then given real numbers y_j, $j = 1, \ldots, N$, there exists a unique polynomial $p \in \pi_{m-1}(\mathbb{R}^2)$ such that $p(x^j) = y_j$.*

Proof. Let $x^{i,j}$ be the intersection point of ℓ_i, ℓ_j. It is easily verified that for $x \in \mathbb{R}^2$, the polynomial

$$p(x) = \sum_{k=1}^{N} y_k \frac{\prod_{r \neq i,j} \ell_k(x)}{\prod_{r \neq i,j} \ell_k(x^{i,j})}$$

satisfies the conclusion of the theorem. □

Corollary 5.3 *Given a convex polygon P, the lines $\ell_0, \ldots, \ell_m$ can be constructed so that x^j, $j = 1, \ldots, \binom{m}{2}$ lie on the interior of P.*

Not only does Theorem 5.2 tell us how to build our interpolants but it also provides the evaluation points for each region.

Thus, our bivariate generalization of Grandine's scheme is summarized in the following algorithm:

Algorithm 5.4 *Given $X = \{x^0, \ldots, x^n\} \subset \mathbb{R}^s$, $n > s$, $vol_s[X] > 0$, and $x \in \mathbb{R}^2$, this algorithm returns the value of $M(x|X)$. Let N denote the number of regions formed by the points in X.*

```
For k from 1 to N do
     Find the interpolation points x^{i,j}
     Use Algorithm 4.3 to evaluate M(x^{i,j}|X)
     Determine the polynomial piece for region k
Determine the region that houses x and evaluate the corresponding polynomial.
```

We conclude the paper by using Algorithm 5.4 to evaluate the splines given in Examples 3.4-3.7 in Section 3.

Example 5.5 *Let $n = 3$ and $X = \{(0,0)^T, (0,0)^T, (1,0)^T, (0,1)^T\}$.*

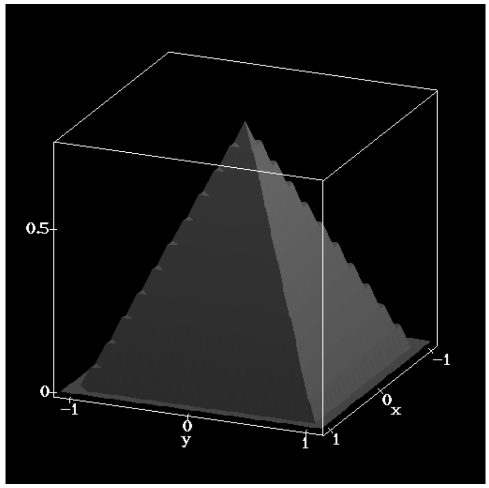

Figure 7. *The domain of the spline from Example 5.5*

Example 5.6 *Let $n = 5$ and $X = \{(0,0)^T, (0,0)^T, (1,0)^T, (1,0)^T, (0,1)^T, (0,1)^T\}$.*

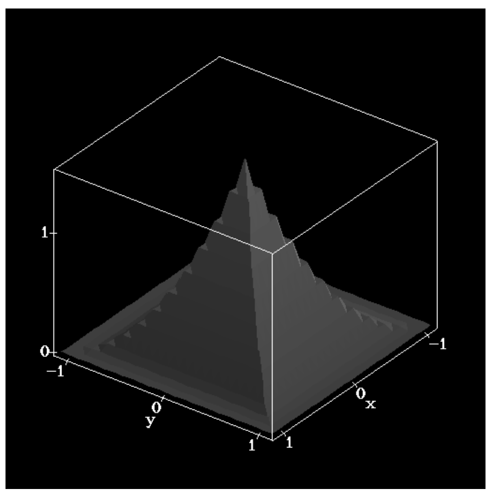

Figure 8. *The domain of the spline from Example 5.6*

Example 5.7 *Let* $n = 3$ *and* $X = \{(1,1)^T, (1,-1)^T, (-1,-1)^T, (-1,1)^T\}$.

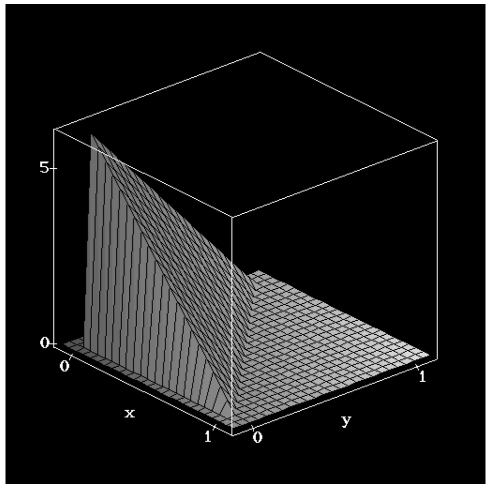

Figure 9. *The domain of the spline from Example 5.7*

Example 5.8 *Let* $n = 3$ *and* $X = \{(1,1)^T, (1,-1)^T, (-1,-1)^T, (-1,1)^T, (0,0)^T\}$.

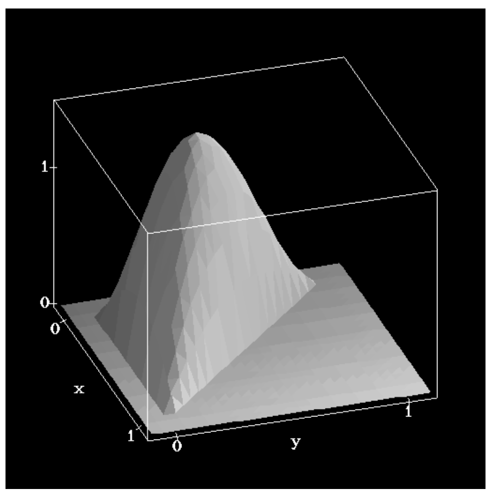

Figure 10. *The domain of the spline from Example 5.8*

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